4.8: Volumes of Solids of Rev (cont'd) and 4.9: Arc Length

Mathematics 3
Lecture 26
Dartmouth College

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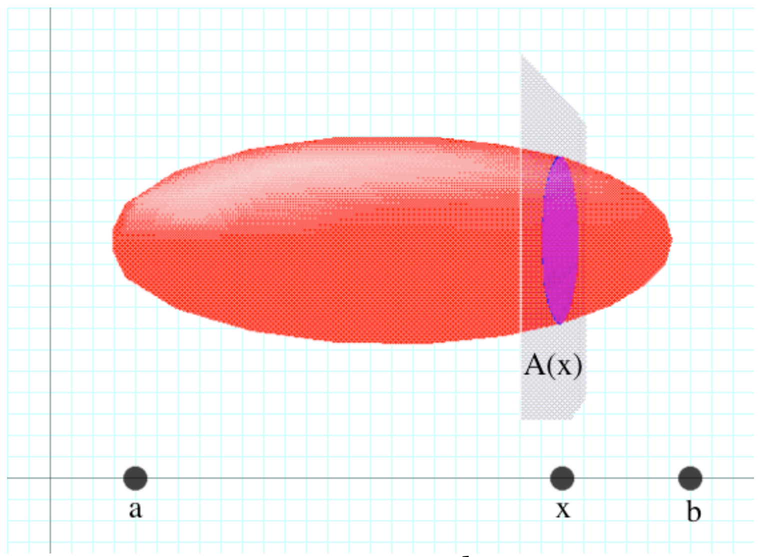
Volume by Slicing (General Solid)

Suppose that a three-dimensional solid lies along the x-axis covering the inteval $[a, b]$ and the cross-sectional area at x is a continuous function, call it it $A(x)$. How do we define/compute it's volume V?



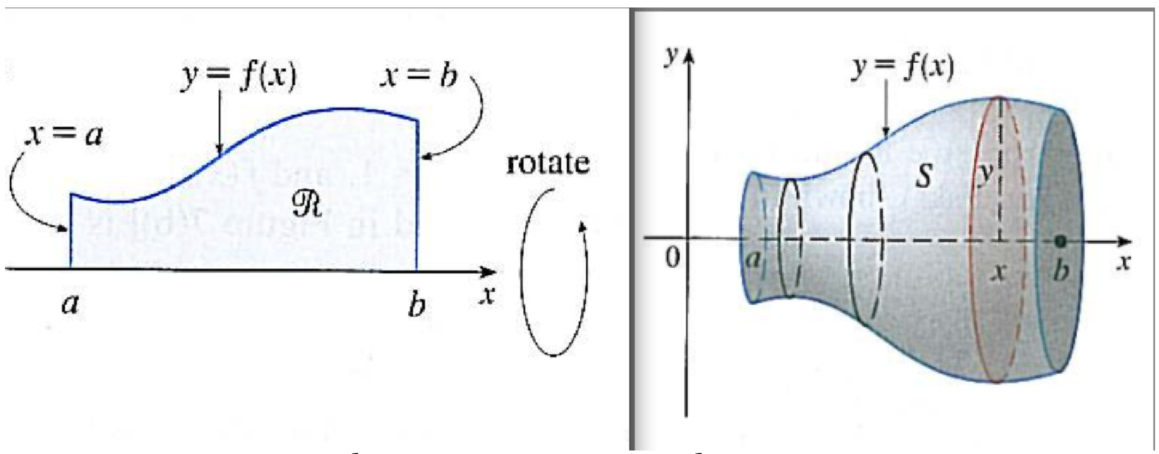
$$\text{Volume } V = \int_a^b A(x)\,dx$$

Solids of Revolution

A 2D region R bounded by $y = f(x)$ over $[a, b]$ is rotated around the x-axis so the area function $A(x)$ is given by:

$$A(x) = \pi r^2 = \pi[f(x)]^2.$$

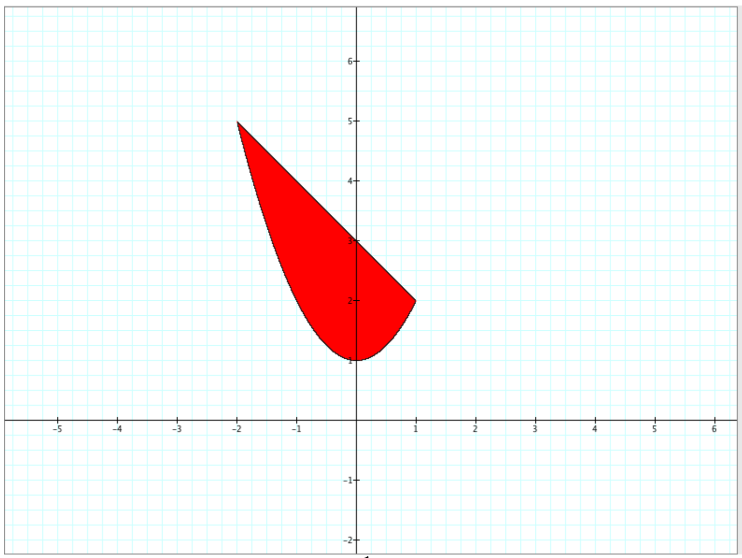
Thus, from Volumes by Slicing, the volume V is the definite integral:

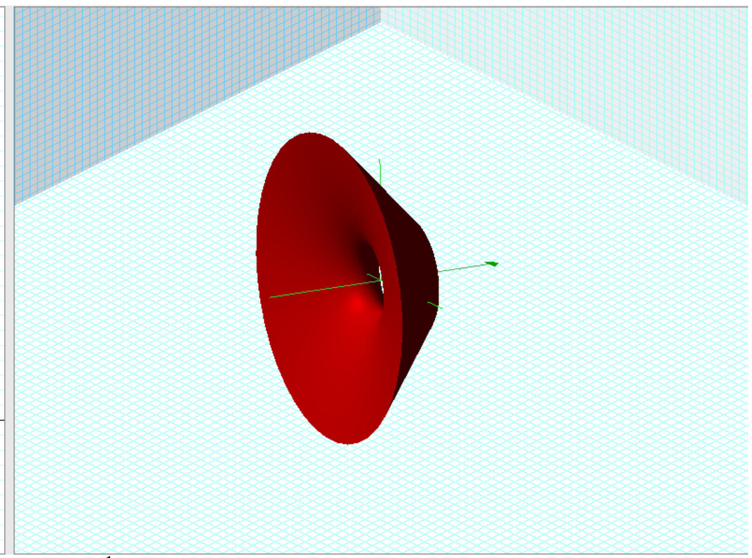


$$V = \int_a^b A(x)\,dx = \int_a^b \pi[f(x)]^2\,dx$$

Example 1

A loudspeaker is to be constructed as the solid of revolution generated by revolving the area between the curves $y = x^2 + 1$ and $y = -x + 3$ around the x-axis. Find the volume of the loudspeaker.





$$V = V_{outer} - V_{inner} = \int_{-2}^{1} \pi(-x+3)^2\,dx - \int_{-2}^{1} \pi(x^2+1)^2\,dx = \pi(30 - \frac{33}{5}) = \frac{117}{5}\pi$$

Real World Problem: Length of Power Lines

How do we compute the length of a power line suspended between two poles?

This is an important problem for power companies and utilities who want to minimize the costs of laying and replacing power lines, especially after winter storms...

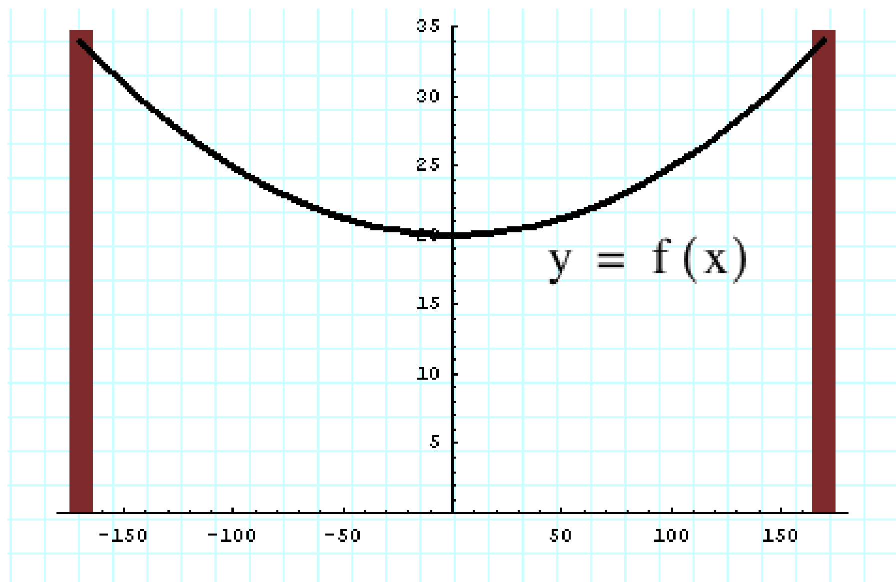
PSNH has more than 13,000 miles of power lines in the state of NH to maintain.

NB: Power line cables cost $\$10$ per foot $=$ $52,800 per mile...

NB: In a blizzard on Thanksgiving weekend 2005 in South Dakota, 10,000 miles of power lines were knocked down!

Mathematical Model: Length of Power Lines

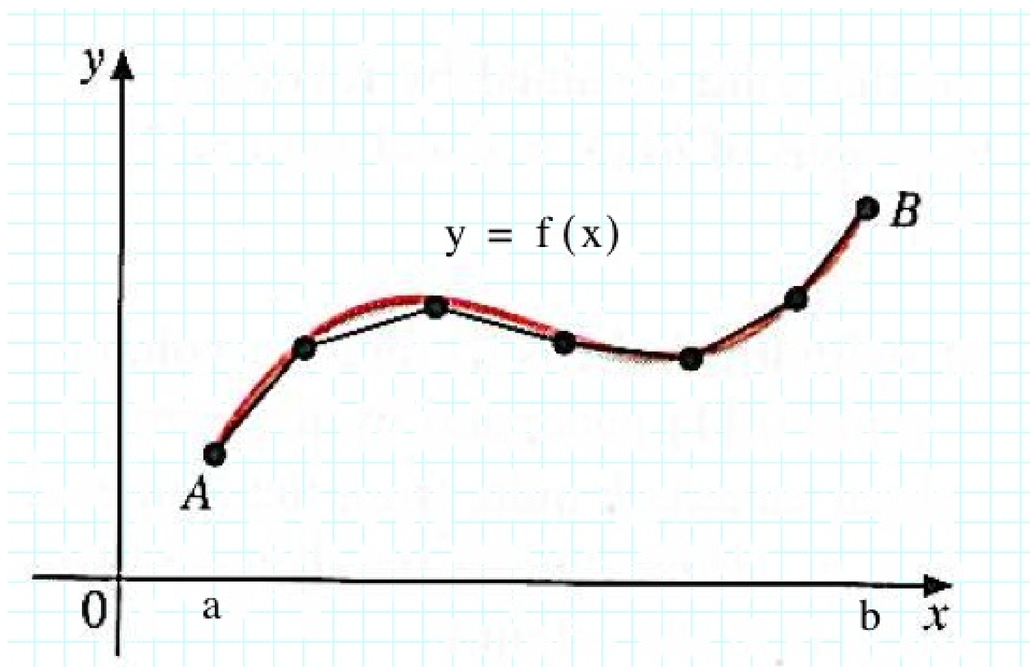
By choosing the obvious coordinates, we can model the power line curve as the graph of a function $y = f(x)$ over an interval $[a, b]$.



Thus, if we can **compute** the length of the graph (= "arc length") of a function $y = f(x)$ over an interval $[a, b]$, we can solve the power line problem.

Approximate Solution to the Arc Length

We will assume that $y = f(x)$ is a continuous function on the interval $[a, b]$ and that the derivative $f'(x)$ exists at every point of the open interval (a, b). We will approximate the curve of the graph of f with a polygonal curve whose length is easy to compute...

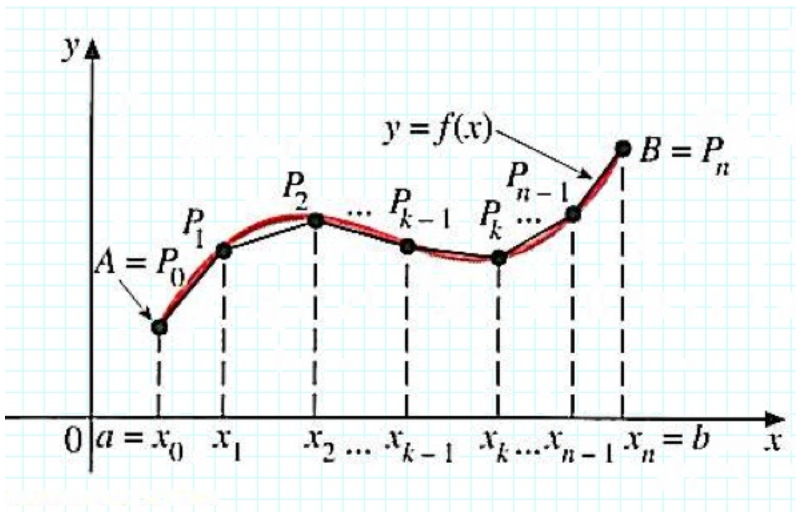


Approximate Solution to the Arc Length

1.) Partition the interval $[a, b]$ into n subintervals:

$$a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_{n-1} \leq x_n = b,$$

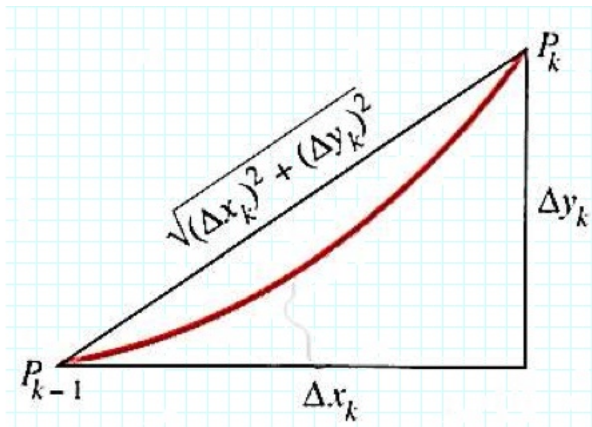
where $x_k = a + k\Delta x$ for each k and with length $\Delta x_k = (x_k - x_{k-1})/n$.
2.) On each subinterval $[x_{k-1}, x_k]$, $1 \leq k \leq n$, connect the endpoints $P_{k-1} = (x_{k-1}, f(x_{k-1}))$ and $P_k = (x_k, f(x_k))$ on the graph of f with straight lines.



Approximate Solution to the Arc Length

3.) The length L_k of the straight-line segment connecting the two points P_{k-1} and P_k is given by the Pythagorean Theorem:

$$(L_k)^2 = (\Delta x_k)^2 + (\Delta y_k)^2.$$

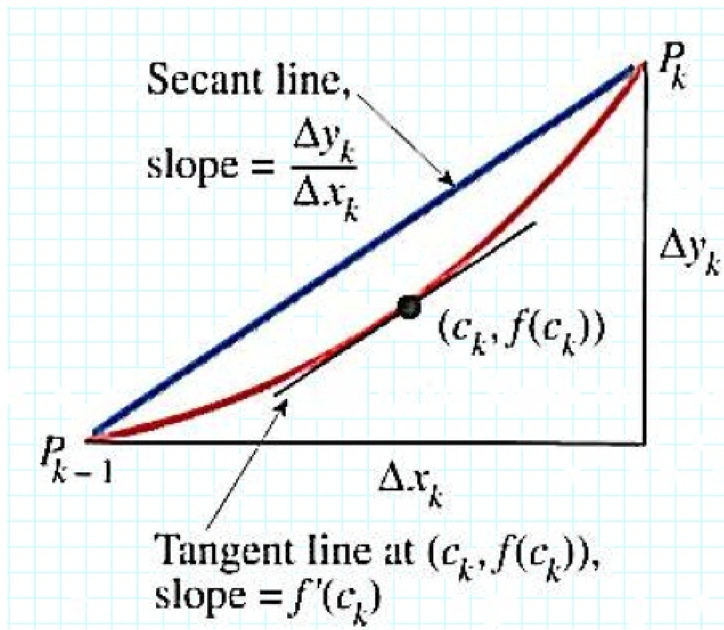


4.) Now use some algebra to rewrite:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}$$

Approximate Solution to the Arc Length

5.) On each subinterval $[x_{k-1}, x_k]$ use the Mean Value Theorem to **choose** a point c_k such that the tangent slope $f'(c_k)$ is equal to the secant slope:



$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta y_k}{\Delta x_k}$$

Riemann (Sums) to the Rescue!

6.) We can replace L_k by the value

$$L_k = \Delta x_k \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2} = \Delta x_k \sqrt{1 + [f'(c_k)]^2}.$$

7.) The (Riemann) sum of the lengths of these line segments provides an approximation to the (actual) length L of the graph of f on $[a, b]$:

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \Delta x_k.$$

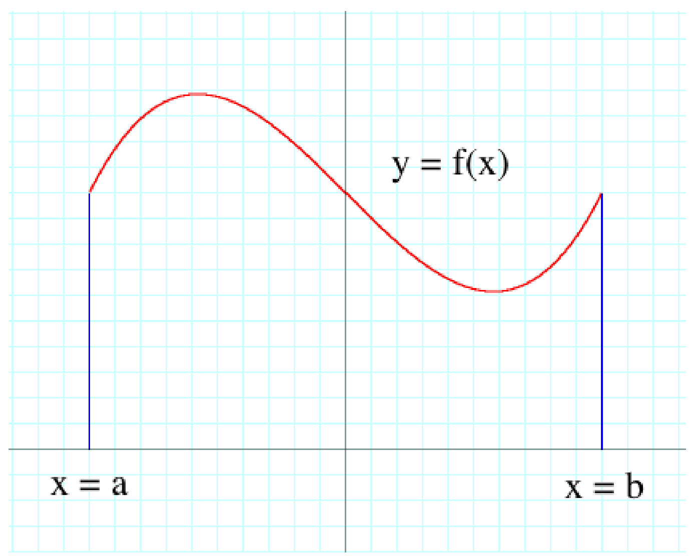
8.) Taking the limit as $\max \Delta x_k \to 0$ ($n \to \infty$), the above approximation approaches the length of the curve L in the limit:

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

The Arc Length Formula

The integral formula to compute the length L of the graph of a (differentiable) function $y = f(x)$ between $x = a$ and $x = b$ is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$



Example 2

Find the length of the arc $y = x^{3/2}$, from $x = 0$ to $x = 1$.

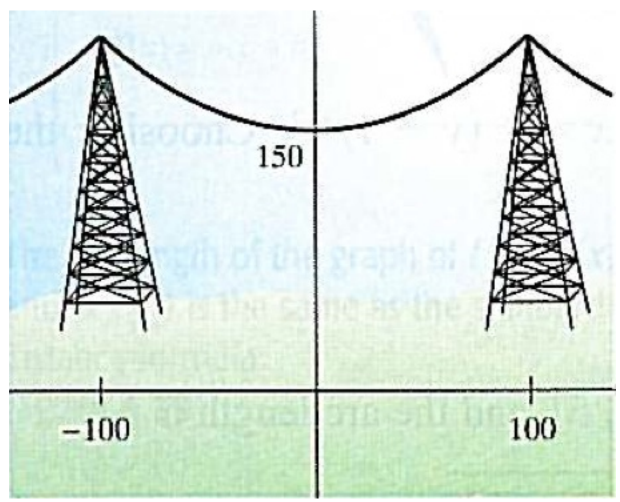
$$L = \int_0^1 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{13^{3/2} - 8}{27}$$

Example 3

An electric power line is hung between two towers that are 200 feet apart. The cable takes the shape of a catenary curve

$$y = 75(e^{x/150} + e^{-x/150}).$$

Find the length of the power line.



Example 3 Calculations

$$
\begin{aligned}
f(x) &= 75(e^{x/150}+e^{-x/150}) \\
f'(x) &= 75\left(\frac{1}{150}e^{x/150}-\frac{1}{150}e^{-x/150}\right)=\frac{1}{2}\left(e^{x/150}-e^{-x/150}\right) \\
(f'(x))^2 &= \left(\frac{1}{2}\left(e^{x/150}-e^{-x/150}\right)\right)^2=\frac{1}{4}(e^{x/75}-2+e^{-x/75}) \\
1+[f'(x)]^2 &= \frac{1}{4}(e^{x/75}+2+e^{-x/75})=\frac{1}{4}\left[e^{x/150}+e^{-x/150}\right]^2
\end{aligned}
$$

$$
\begin{aligned}
L &= \int_a^b \sqrt{1+[f'(x)]^2}\,dx=\int_{-100}^{100}\frac{1}{2}(e^{x/150}+e^{-x/150})\,dx \\
&= \frac{1}{2}\int_{-100}^{100}(e^{x/150}+e^{-x/150})\,dx=\int_0^{100}(e^{x/150}+e^{-x/150})\,dx \\
&= 150(e^{x/150}-e^{-x/150})\Big|_0^{100}=150(e^{2/3}-e^{-2/3})\approx 215 \text{ feet}
\end{aligned}
$$

Example 4

Find the arc length of the graph of

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

on the interval $[\frac{1}{2}, 2]$.

Solution: $f'(x) = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$

$$\begin{aligned} L &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_{1/2}^2 \sqrt{1 + \left[\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right]^2}\, dx \\ &= \int_{1/2}^2 \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\, dx = \int_{1/2}^2 \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx \\ &= \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{x}\right)\Bigg|_{1/2}^2 = \frac{1}{2}\left(\frac{13}{6} + \frac{47}{24}\right) = \frac{33}{16} \end{aligned}$$

Example 5

Find the length of the curve $y = x^4 + \frac{1}{32x^2}$ from x = 1 to x = 2.

$$y' = 4x^3 - \frac{2}{32x^3} = 4x^3 - \frac{1}{16x^3}$$

$$\begin{aligned}
L &= \int_1^2 \sqrt{1 + \left[4x^3 - \frac{1}{16x^3}\right]^2}\, dx \\
&= \int_1^2 \sqrt{1 + 16x^6 - \frac{8}{16} + \frac{1}{256x^6}}\, dx \\
&= \int_1^2 \sqrt{\frac{8}{16} + 16x^6 + \frac{1}{256x^6}}\, dx \\
&= \int_1^2 \sqrt{\left(4x^3 + \frac{1}{16x^3}\right)^2}\, dx \\
&= \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right)\, dx \\
&= \left(x^4 - \frac{1}{32x^2}\right)\Big|_1^2 \\
&= 15 + \frac{3}{128}
\end{aligned}$$