2.5 Continuity and
2.6 Tangent Lines and Their Slope

Mathematics 3
Lecture 7
Dartmouth College

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Limits and Limits at Infinity (cont'd)

Warning: Analyzing limits graphically can be misleading...

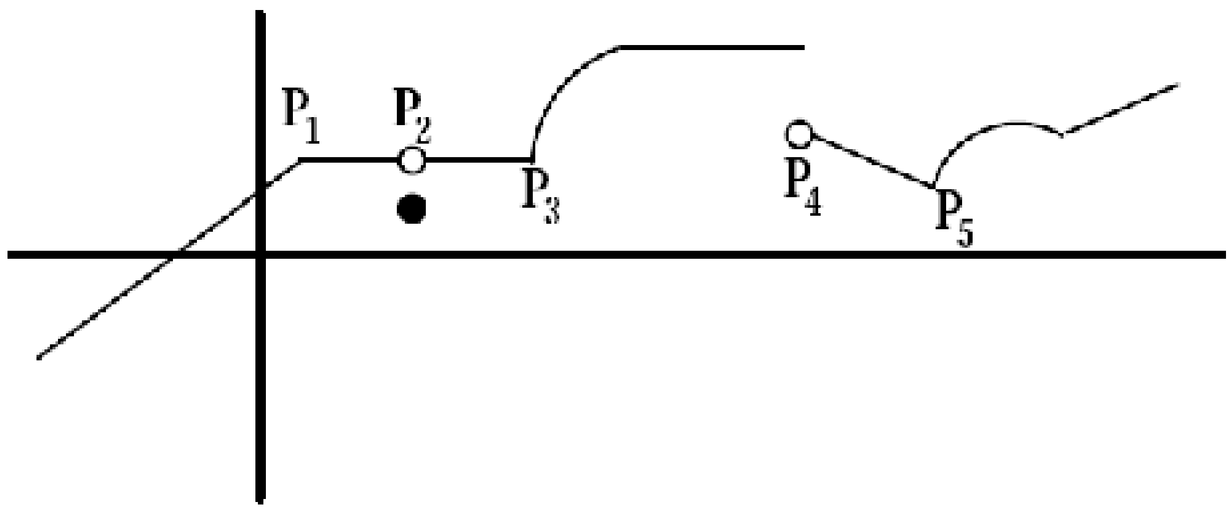
Example 1: Compute the following limits.

a.) $\lim_{x \to 0} \frac{(x^3+8)^{1/3}-2}{x^3}$

b.) $\lim_{x \to \infty} \frac{x+100x^{1.8}+4x^2}{2x^2-x+1}$

Intuition: Continuity of a Function

A function is *continuous* if it's graph "can be drawn without lifting the pencil from the paper."



Note: A *dis*continuity can occur at a *single* point (where we "lift the pencil") so we must define it's opposite: continuity at a point.

Interior Point

An *interior point* of a set S of real numbers is a point that can be enclosed in an open interval that is contained in the set S.

Example 2: Find the interior of the set

$$S = (-\infty, 3] \cup \{5, 6\} \cup (8, 10].$$

Definition: (Dis)continuity at an Interior Point

- A function is **continuous at an interior point** c of its domain if
$$\lim_{x \to c} f(x) = f(c).$$

- If it is not continuous there, i.e., if either the limit does not exist or is not equal to $f(c)$ we will say that the function is **discontinuous at the point** c.

Note: Requirements for continuity at a point

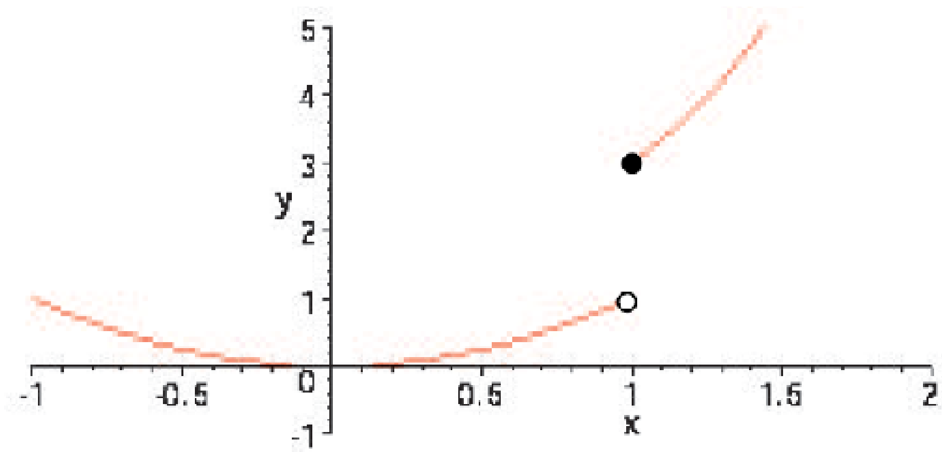
- The function f is defined at the point $x = c$,
- The point $x = c$ is an interior point of the domain of f,
- $\lim_{x \to c} f(x)$ exists, say it equals L, and
- $L = f(c)$.

Example 3

Is the function

$$f(x) = \begin{cases} x^2, & x < 1 \\ x^3 + 2, & x \geq 1 \end{cases}$$

continuous at $x = 1$?



Right Continuity and Left Continuity at a Point

- A function f is *right continuous at a point* c if it is defined on an interval $[c, d]$ lying to the right of c and if

$$\lim_{x \to c^+} f(x) = f(c).$$

- Similarly it is *left continuous at* c if it is defined on an interval $[d, c]$ lying to the left of c and if

$$\lim_{x \to c^-} f(x) = f(c).$$

Formal Definition of Continuity at a Point

Def: A function f is *continuous at a point* $x = c$ if c is in the domain of f and if:

1. $x = c$ is an interior point of the domain and $\lim_{x \to c} f(x) = f(c)$.

2. $x = c$ is not an interior point of the domain but is an endpoint of the domain, then f must be right or left continuous at $x = c$, i.e., the right/left hand limit

$$\lim_{x \to c^{\pm}} f(x) = f(c)$$

as appropriate.

Even More Definitions...

- A function f is said to be a *continuous function* if it is continuous **at every point of its domain**.

- A *point of discontinuity* of a function f is a point in the domain of f at which the function is NOT continuous.

Continuous Function Facts

All of the following types of “elementary” functions:

- Polynomials,
- Rational functions,
- Trigonometric functions,
- The absolute value function, and
- Exponential and logarithmic functions

are continuous wherever they are defined, i.e., on their **maximal** domains of definition!

Example 4: Continuous Extensions of Discontinuous Functions

- The rational function $f(x) = \frac{x^2-2x-3}{3-x}$ is a continuous function.
- The domain is all real numbers *except* $x = 3$.
- $\lim_{x \to 3} f(x) = -4$ exists. Why??

It has a *continuous extension*

$$F(x) \quad = \quad \begin{cases} f(x) & \textit{if } x \neq 3 \\ -4 & \textit{if } x = 3 \end{cases}$$

that is continuous on the whole real line!

Example 5: Removing a discontinuity

The function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is discontinuous at $x = 0$.

We can "remove" the discontinuity by redefining the value of f at 0. How?

Definition: Removable Discontinuity

- If c is a point of discontinuity of a function f, and if

$$\lim_{x \to c} f(x) = L$$

  exists, then c is called a *removable discontinuity*. The discontinuity is removed by (re)defining $f(c) = L$.

- If f is not defined at c but $\lim_{x \to c} f(x) = L$ exists, then f has a continuous extension to $x = c$ by defining $f(c) = L$.

Example 6

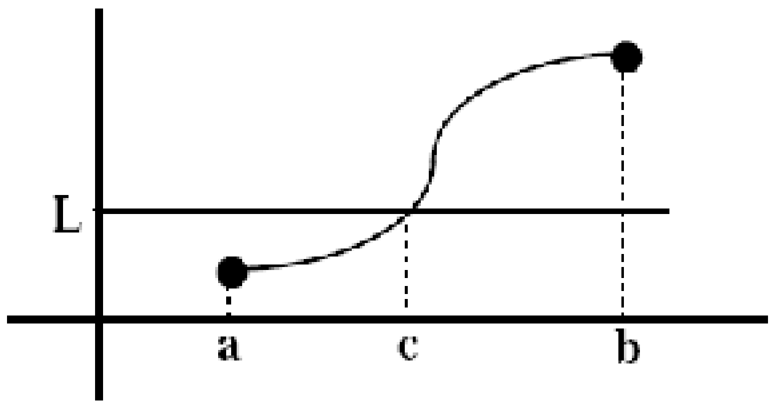
Suppose that $f(x)$ is defined piecewise as

$$f(x) = \begin{cases} -x^2 + 1 & x < 2 \\ x + k & x > 2 \end{cases}$$

Let us find a value of the constant k such that f has a continuous extension to $x = 2$.

The Intermediate Value Theorem

If a function f is continuous on a closed interval $[a, b]$, and if $f(a) < L < f(b)$ (or $f(a) > L > f(b)$), then there exists a point c in the interval $[a, b]$ such that $f(c) = L$.



NB: This result may be intuitively obvious, but requires very advanced mathematical techniques to prove! (Math 35/54/63)

Example 7: Using the IVT to show solutions to equations exist...

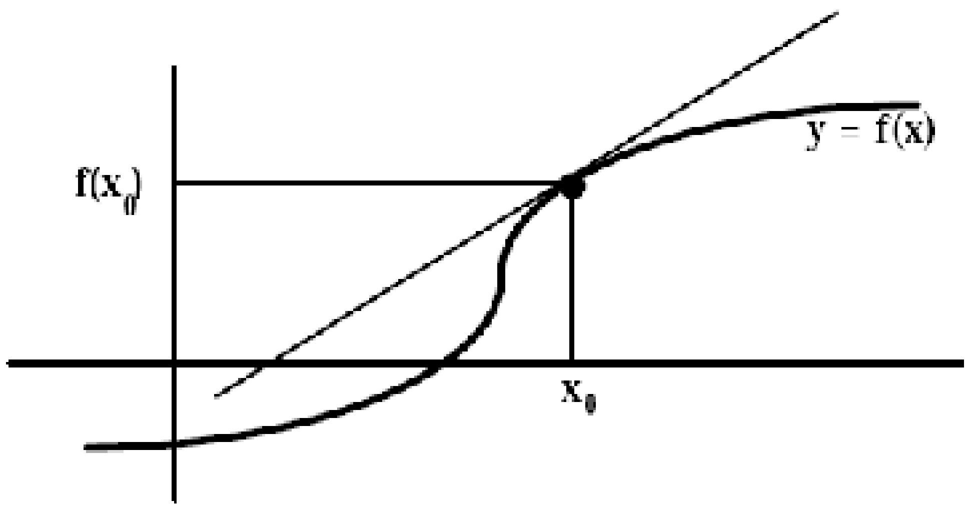
Show that the polynomial equation

$$x^5 - 3x + 1 = 0$$

has a solution in the interval $[0, 1]$.

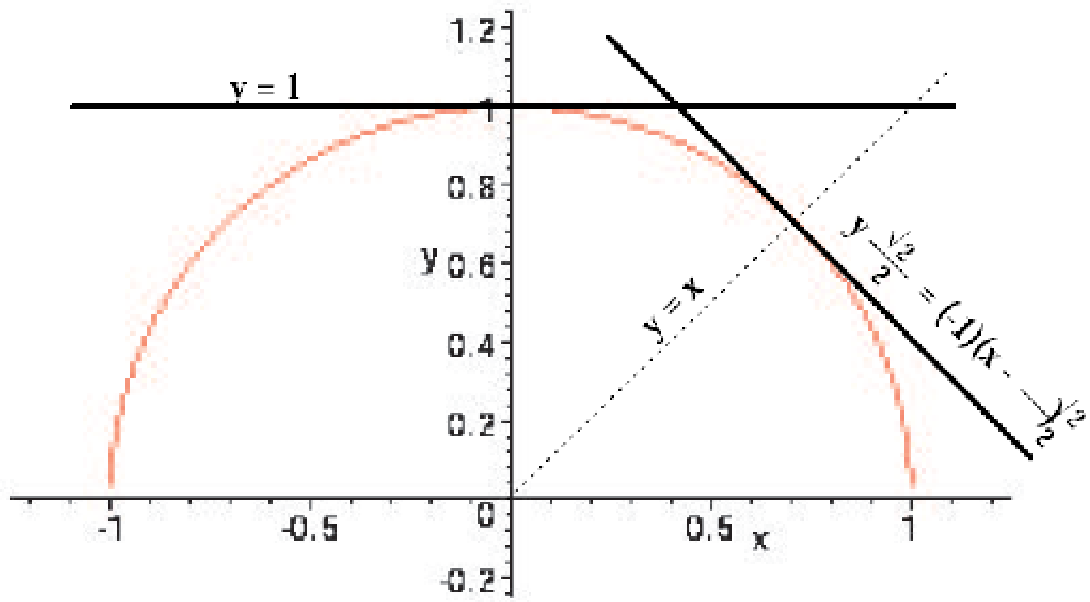
2.6: Tangent Lines and Their Slope

The Tangent Line Problem: Given a function $y = f(x)$ defined in an open interval and a point x_0 in the interval, define the tangent line at the point $(x_0, f(x_0))$ on the graph of f.



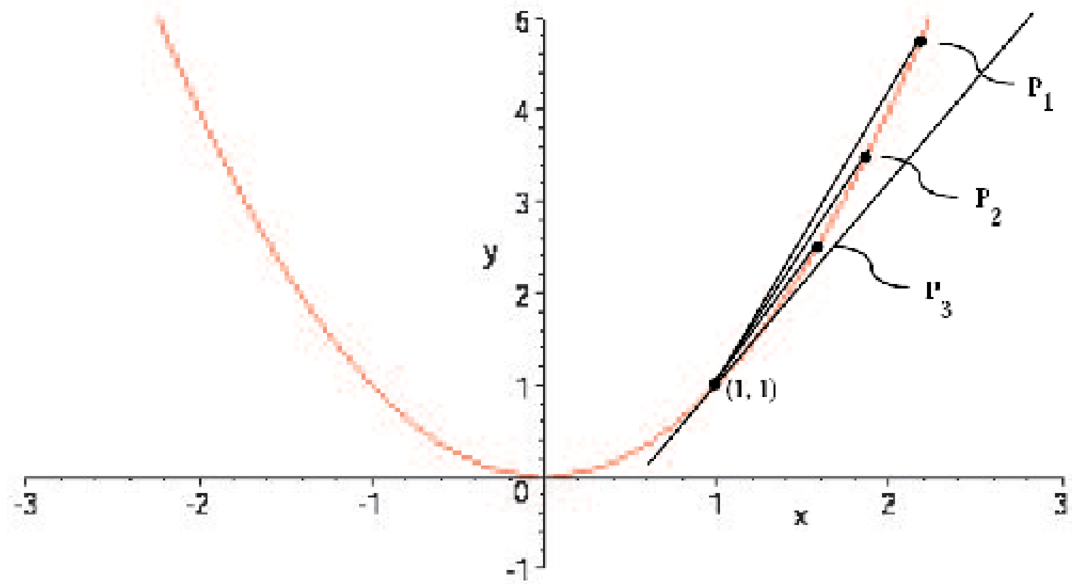
Example 8

Find the equations of the tangent lines to the graph of $f(x) = \sqrt{1 - x^2}$ at the points $(0, 1)$ and $(\frac{\sqrt{2}}{2}.\frac{\sqrt{2}}{2})$.



Example 9

Let $f(x) = x^2$. Find the equation of the tangent line at any point (x_0, x_0^2) on the graph of f.



Definition: Slope of the Tangent Line

Given a function f and a point x_0 in its domain, the slope of the tangent line at the point $(x_0, f(x_0))$ (or at x_0) on the graph of f is

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

IF THIS LIMIT EXISTS...

If this limit does NOT exist, then f does NOT have a (slope of a) tangent line at x_0.

NB: The limit above is equivalent to (by substituting $x = x_0 + h$)

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Example 10

Given $f(x) = \sqrt{x}$, find the equation of the tangent line at $x = 4$.