2.7: The Derivative

Mathematics 3
Lecture 8 (x-hour)
Dartmouth College

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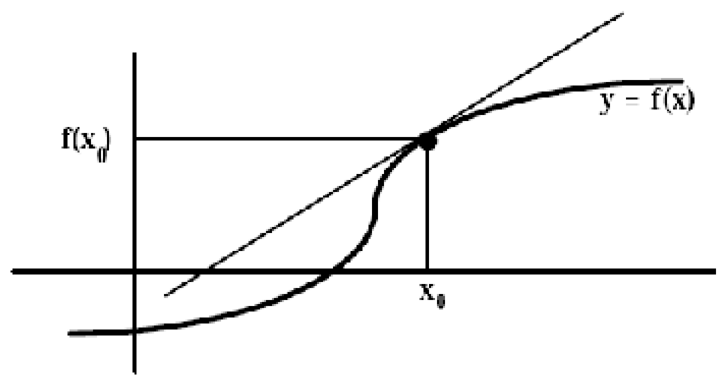
Recall: Slope of the Tangent Line

Given a function f and a point x_0 in its domain, the slope of the tangent line at the point $(x_0, f(x_0))$ (or at x_0) on the graph of f is

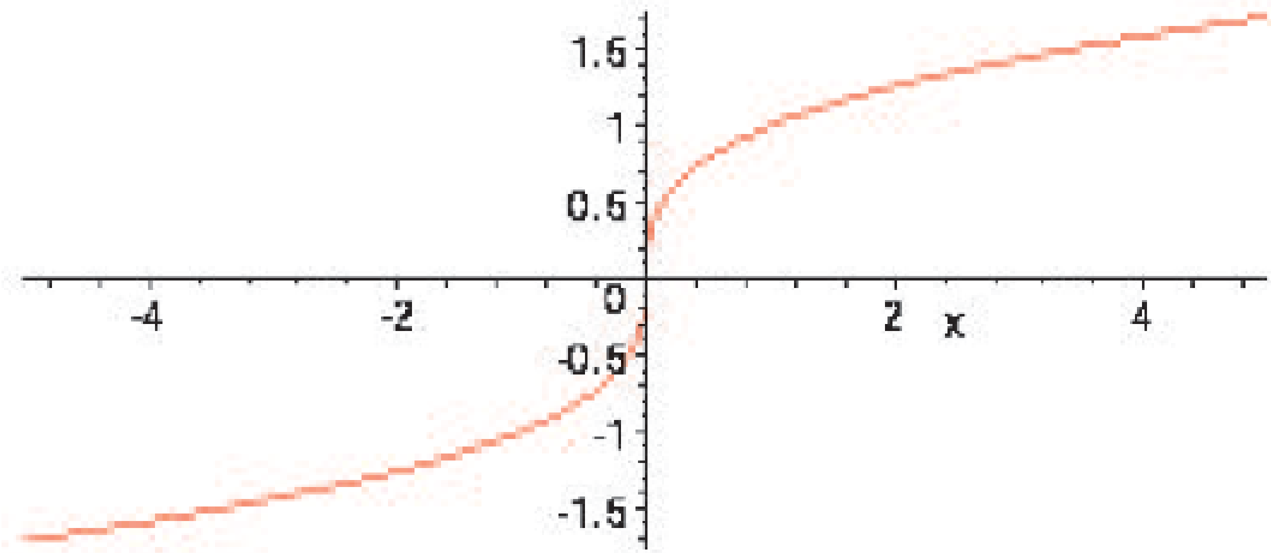
$$m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

IF THIS LIMIT EXISTS... If this limit does NOT exist, then f does NOT have a (slope of a) tangent line at x_0.



Example 0

Does the function $f(x) = x^{1/3}$ have a tangent line at $x = 0$?



Definition of the Derivative Function

- The derivative of a function f is a new function f' (read "f prime") defined by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$
This function (which is derived from f) is only defined at those values x where the limit exists. It may have a smaller domain than the domain of f.

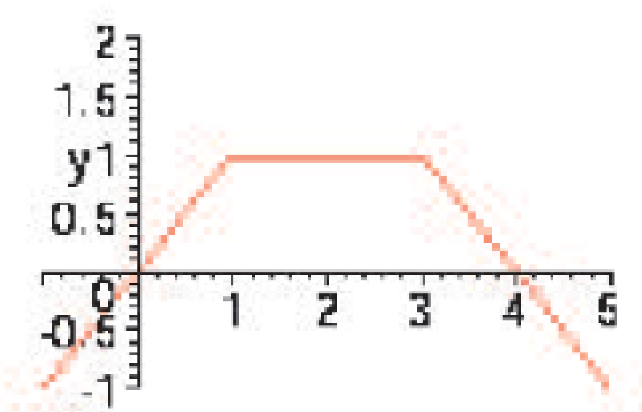
- We will say that a function f is differentiable at a point $x = a$ if the derivative function f' exists at a. Note that the number $f'(a)$ is the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$. The tangent line to the graph at $(a, f(a))$ is:

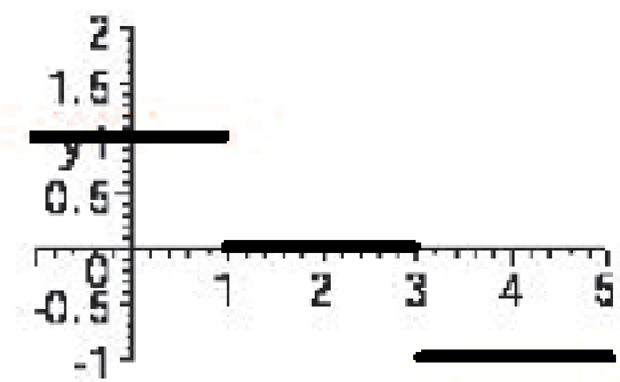
$$\boxed{y - f(a) = f'(a)(x - a)}$$

Example 1: Finding the derivative graphically

Suppose we consider the piecewise defined function

$$f(x) = \begin{cases} x, & x \le 1 \\ 1, & 1 < x < 3 \\ -x + 4, & x \ge 3 \end{cases}$$





It's derivative function is:

$$f'(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & x < 3 \end{cases}$$

Note that the derivative function is NOT defined at $x = 1, 3$.

Example 2: Derivative of Constant Functions

For a constant function

$$f(x) = k,$$

where k is a constant, what is the derivative function $f'(x)$?

Example 3: Derivative of Linear Functions

For a linear function,

$$f(x) = ax + b,$$

where a, b are constants, what is the derivative function $f'(x)$?

Example 4: The derivative of x^2

For $f(x) = x^2$, we have (using Example 9 from yesterday) that

$$f'(x) = 2x$$

since the tangent line at (x, x^2) has slope $2x$.

Example 5: The derivative of x^3

Use the formula

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

to find the derivative of $f(x) = x^3$ at x_0.

Example 6: The derivative of $1/x$

For $f(x) = \frac{1}{x}$, show that

$$f'(x) = -\frac{1}{x^2}$$

Question: Does anyone notice a pattern here...?

Example 7: The derivative of $\sqrt{x}$

For $f(x) = \sqrt{x} = x^{1/2}$, we have (see yesterday's Example 10)

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-\frac{1}{2}}$$

Theorem 1: The Power Rule

Suppose that $f(x) = x^r$, where $r \neq 0$ is a real number. Then

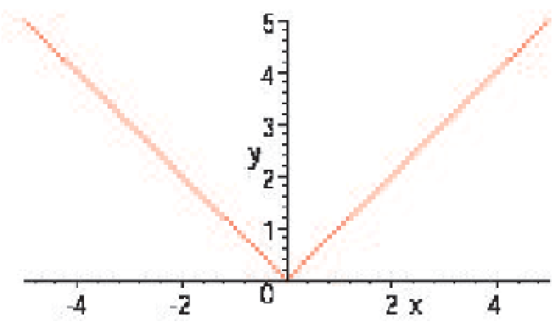
$$f'(x) = rx^{r-1}.$$

Example 8

Find the slope-intercept equation $y = mx + b$ of the tangent line to the graph of $f(x) = x^{4/3}$ at the point where $x = 8$. From page 3, the tangent line has the basic equation

$$y = f(8) + f'(8)(x - 8).$$

Example 9: Find the derivative of $f(x) = |x|$



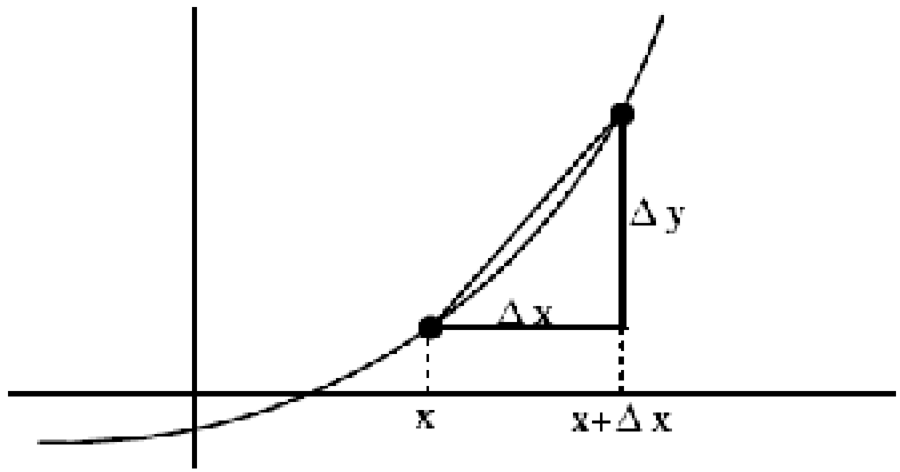
$$
\begin{aligned}
\lim_{h\to 0^+} \frac{|0+h|-0}{h} &= 1 \\
\lim_{h\to 0^-} \frac{|0+h|-0}{h} &= -1
\end{aligned}
$$

Notations for the Derivative Function

For a function $y = f(x)$ we can denote the derivative function by:

$$y' = D_x y = \frac{dy}{dx} = \frac{d}{dx} f(x) = \frac{df}{dx} = f'(x).$$

The Leibniz notation $\frac{dy}{dx}$



Substituting $\Delta x = h$ in the definition of the derivative (page 2):

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

The derivative function is NOT a fraction, but a LIMIT of fractions (i.e., difference quotients).

Higher Order Derivatives

- When we differentiate a function $f(x)$ we obtain a new function $f'(x)$.

- The derivative is again a candidate for differentiation, and we call its derivative the second derivative $f''(x)$ of $f(x)$.

- So long as the derivatives exist we can continue this process to obtain a succession of **higher derivatives**.

$$y'' = f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = D_x^2 y = D_x^2 f(x).$$

The nth derivative, where n is a positive integer

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n} = \frac{d^n}{dx^n} f(x) = D_x^n y = D_x^n f(x).$$