Math 3 - Day 1 - WARMUP

Calculate the given functions at the given $x$-values, and then plot the corresponding points.

$f(x) = x^2$

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  -2 & \\
  -1 & \\
  0 & \\
  1 & \\
  2 & \\
\end{array}
\]

\[y\]

\[x\]

\[f(x) = (x + 1)^2\]

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  -2 & \\
  -1 & \\
  0 & \\
  1 & \\
\end{array}
\]

\[y\]

\[x\]

\[f(x) = x^2 + 1\]

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  -2 & \\
  -1 & \\
  0 & \\
  1 & \\
  2 & \\
\end{array}
\]

\[y\]

\[x\]
Welcome to Math 3!

http://www.math.dartmouth.edu/~m3f13/

Me:

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Office hours: Wednesday 2-3, Thursday 10-12, or by appointment
Extra credit: Your first visit to office hours after this week

Tutorials:

7:00 to 9:00 pm on Sunday, Tuesday, and Thursday  
Begins tomorrow in 008 Kemeny.

Highlights:

Homework: 15% of your grade  
Tips: 
All done on WeBWorK – due at 8:00 a.m two classes later (answers up at 12pm)  
When possible, use exact answers like “2^6 – 1” instead of “63”.  
Wrong submissions don’t count against you, unless otherwise stated.  
   Pro: as many tries as you need  
   Con: computers are fussy

Midterms: 25% × 2  
Part multiple choice, part long-answer.  
Dates:  
Midterm 1: Wed 10/16, 7–9 pm  
Midterm 2: Wed 11/6, 7–9 pm

Final: 35%  
All multiple choice. Cumulative.  
Date: 11/22, 11:30 am
Functions and their graphs

Simplest functions: Lines!

Two points define a line!

Slope: \[ m = \frac{y_2 - y_1}{x_2 - x_1} \] (rise/run)

Point-slope form: \[ y - y_1 = m(x - x_1) \] (good for writing down lines)

Slope-intercept form: \[ y = mx + b \] (good for graphing)

General form: \[ Ax + By + C = 0 \] (accounts for \( \infty \) slope)
Simplest functions: Lines!

Example:

Slope: \[ m = \frac{3.4 - 1.8}{3 - 1} = 0.8 \] (rise/run)

Point-slope form: \[ y - 1.8 = 0.8 \cdot (x - 1) \] (good for writing down lines)

Slope-intercept form: \[ y = 0.8 \cdot x + 1 \] (good for graphing)

General form: \[ 0.8 \cdot x + y - 1 = 0 \] (accounts for $\infty$ slope)

Lines: Special cases

Constant functions \((m = 0)\)

Vertical lines \((m = \infty)\)

Parallel lines \((m_1 = m_2)\)

Perpendicular lines \((m_1 = -1/m_2)\)
Other good functions to know: polynomials.

\[ y = a_0 + a_1x + \cdots + a_nx^n \]

\((n\) is the degree\)

The basics (know these graphs!)

\( n = 0: \) \hspace{1cm} \( n = 1: \) \hspace{1cm} \( n = 2: \)

constants \hspace{1cm} lines \hspace{1cm} parabolas

\[
\begin{align*}
y &= x^2 \\
y &= x^3 \\
y &= x^4 \\
y &= x^5
\end{align*}
\]

Other good functions to know: rationals.

\[ y = \frac{a_0 + a_1x + \cdots + a_nx^n}{b_0 + b_1x + \cdots + b_mx^m} \]

The basics (know these graphs!)

\[
\begin{align*}
y &= \frac{1}{x} \\
y &= \frac{1}{x^2} \\
y &= \frac{1}{x^3} \\
y &= \frac{1}{x^4}
\end{align*}
\]
Other powers: $y = x^a$.

The basics (know these graphs!)

\begin{align*}
y &= x^{1/2} = \sqrt{x} \\
y &= x^{1/3} = \sqrt[3]{x}
\end{align*}
New functions from old

Graph of $y = f(x)$:

Graph of $y = f(x + 2)$ (left shift):

Graph of $y = f(x) + 1$ (up shift):

Graph of $y = f(2x)$ (horizontal squeeze):

Graph of $y = 4 \cdot f(x)$ (vertical dilation):

Graph of $y = f(-x)$ (vertical reflection):

Graph of $y = -f(x)$ (horizontal reflection):

Graph of $-y = f(-x)$ (180° rotation):

Graph of $x = f(y)$ (flip over $y = x$):
Ex: Transform the graph of \( f(x) \) into the graph of \(- f \left( \frac{1}{2} (x + 1) \right) + 2:\)

The *domain* of a function \( f \) is the set of \( x \) over which \( f(x) \) is defined.

The *range* of a function \( f \) is the set of \( y \) which satisfy \( y = f(x) \) for some \( x \).
Symmetries

A function $f(x)$ is **even** if it satisfies

$$f(-x) = f(x)$$

A function $f(x)$ is **odd** if it satisfies

$$f(-x) = -f(x)$$

**Examples: Even, odd, or neither?**

(a) 

(b) 

(c) 

(d) $f(x) = \frac{x^3 + x}{x + \frac{1}{x}}$

(for this one: actually plug in $-x$ and see what happens algebraically)
A graph is a graph of a *function* if for every \( x \) in its domain, there is exactly one \( y \) on the graph which is mapped to by that \( x \):

**Function:**

![Function Graph](image1.png)

**Not a function:**

![Not a Function Graph](image2.png)

A function is additionally *one-to-one* if for every \( y \), there is at most one \( x \) which maps to that \( y \).

**A one-to-one function:**

![One-to-One Function Graph](image3.png)

### Inverse functions

Let \( f \) be a one-to-one function.

If \( g \) is a function satisfying

\[
f(g(x)) = g(f(x)) = x
\]

then \( g \) is the *inverse function* of \( f \). Write \( g(x) = f^{-1}(x) \).

**Example:** If \( f(x) = x^3 \), then \( f^{-1}(x) = \sqrt[3]{x} \).

To calculate \( f^{-1}(x) \), set \( f(y) = x \) and solve for \( y \). Then \( y = f^{-1}(x) \).

**Example:** If \( f(x) = \frac{2x}{x-1} \),

\[
solve x = \frac{2y}{y-1} \text{ for } y \text{ to get } y = \frac{x}{x-2}.
\]

So \( f^{-1}(x) = \frac{x}{x-2} \).

To get the graph of \( f^{-1}(x) \), flip the graph of \( f(x) \) over the line \( y = x \).
Pair up graphs with their inverses:

A

B

C

D
One way to build functions is by composition, i.e. plugging one function into another. If $f(x)$ and $g(x)$ are functions, then for whatever $x$ for which $g(x)$ is in the domain of $f(x)$, then we can write

$$(f \circ g)(x) = f(g(x)).$$

For example, if $f(x) = \frac{x+1}{3x-2}$ and $g(x) = \sqrt{x}$, then

$$(f \circ g)(x) = \frac{\sqrt{x} + 1}{3\sqrt{x} - 2} \quad (\text{for } x \neq \pm(2/3)^2)$$

and

$$(g \circ f)(x) = \sqrt{\frac{x + 1}{3x - 2}} \quad (\text{whenever } \frac{x + 1}{3x - 2} \geq 0).$$

1. Let $f(x) = \frac{x+1}{3x-2}$ and $g(x) = \frac{1}{x}$.

   (a) Calculate $(f \circ g)(x)$ and $(g \circ f)(x)$.

   (b) What is the domain of $(g \circ f)(x)$?  
      [hint: Careful!  The domain of $(g \circ f)(x)$ is the set of $x$’s which satisfy both (1) $f(x)$ exists, and (2) $(g \circ f)(x)$ exists.]
2. Let $f = \frac{x+1}{3x-2}$

(a) Calculate $f^{-1}(x)$.

(b) Check your answer to #1 by explicitly calculating $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$ (you should get $x$ both times).
(c) If \((f \circ g)(x) = x + 2\), what is \(g(x)\)?

[Hint: since \((f \circ g)(x) = f(g(x)) = x + 2\), we know

\[
g(x) = f^{-1}(f(g(x)) = f^{-1}(x + 2).
\]
Answers:

1. (a) \((f \circ g)(x) = \frac{x+1}{2x^2-2}\), \((g \circ f)(x) = \frac{3x-2}{x+1}\)
   
   (b) All \(x \neq 2/3, -1\), i.e. \((-\infty, -1) \cup (-1, 2/3) \cup (2/3, \infty)\)

2. (a) \(f^{-1}(x) = \frac{2x+1}{3x-1}\)
   
   (b) (calculation)
   
   (c) \(g(x) = \frac{2x+5}{3x+5}\)