Exponential and Logarithm Functions
The Basics

If $n$ and $m$ are positive integers...

$$a^n = a \cdot a \cdots a \quad \text{(WeBWoRK: } a^n \text{ or } a**n)$$

Some identities:

Examples:

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2$$
The Basics

If $n$ and $m$ are positive integers...

$$a^n = a \cdot a \cdots \cdot a$$

(WeBWoRK: $a^n$ or $a \cdot^n$)

Some identities:

Examples:

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$
$$2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^8$$
The Basics

If \( n \) and \( m \) are positive integers...

\[
a^n = a \cdot a \cdot \cdots \cdot a
\]

(WeBWoRK: \( a^n \) or \( a \cdot \cdot \cdot n \))

Some identities:

\[
a^n \cdot a^m = a^{n+m}
\]

Examples:

\[
2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2
\]

\[
2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^8
\]
The Basics

If $n$ and $m$ are positive integers...

$$a^n = \underbrace{a \cdot a \cdots a}_n$$ (WeBWoRK: $a^n$ or $a \cdot a \cdot \cdots \cdot a$)

Some identities:

$$a^n \cdot a^m = a^{n+m}$$

Examples:

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$
$$2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^8$$
$$\left(2^3\right)^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^{15}$$
The Basics

If \( n \) and \( m \) are positive integers...

\[
a^n = a \cdot a \cdot \cdots \cdot a
\]

(WeBWoRK: \( a^n \) or \( a \star \star n \))

Some identities:

\[
a^n \star a^m = a^{n+m}
(a^n)^m = a^{n \star m}
\]

Examples:

\[
2^5 = 2 \star 2 \star 2 \star 2 \star 2
\]

\[
2^5 \star 2^3 = (2 \star 2 \star 2 \star 2 \star 2) \star (2 \star 2 \star 2) = 2^8
\]

\[
(2^3)^5 = (2 \star 2 \star 2) \star (2 \star 2 \star 2) \star (2 \star 2 \star 2) \star (2 \star 2 \star 2) \star (2 \star 2 \star 2) = 2^{15}
\]
The Basics

If $n$ and $m$ are positive integers...

$$a^n = a \cdot a \cdot \cdots \cdot a \quad (\text{WeBWoRK: } a^n \text{ or } a * * n)$$

Some identities:

$$a^n \cdot a^m = a^{n+m} \quad (a^n)^m = a^{n\cdot m}$$

(Notice: $a^{mn}$ means $a^{(mn)}$, since $(a^m)^n$ can be written another way)

Examples:

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$$
$$2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^8$$

$$(2^3)^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^{15}$$

$$2^{3^5} = 2^{243} >> (2^3)^5 = 2^{15}$$
The Basics

If \( n \) and \( m \) are positive integers...

\[
a^n = a \cdot a \cdot \ldots \cdot a
\]

(WeBWoRK: \( a^n \) or \( a \cdot \cdot \cdot n \))

Some identities:

\[
a^n \cdot a^m = a^{n+m} \quad (a^n)^m = a^{n\cdot m}
\]

(Notice: \( a^{m^n} \) means \( a^{(m^n)} \), since \( (a^m)^n \) can be written another way)

Examples:

\[
2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\
2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^8 \\
(2^3)^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^{15} \\
2^{3^5} = 2^{243} \gg (2^3)^5 = 2^{15} \\
2^3 \cdot 5^3 = (2 \cdot 2 \cdot 2) \cdot (5 \cdot 5 \cdot 5) = (2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5) = (2 \cdot 5)^3
\]
The Basics

If $n$ and $m$ are positive integers...

$$a^n = a \cdot a \cdot \cdots \cdot a$$  (WeBWoRK: $a^n$ or $a \ast \ast n$)

Some identities:

$$a^n \cdot a^m = a^{n+m}$$  \hspace{1cm}  \( (a^n)^m = a^{n \cdot m} \)

(Notice: $a^{m^n}$ means $a^{(m^n)}$, since $(a^m)^n$ can be written another way)

$$a^n \cdot b^n = (a \cdot b)^n$$

Examples:

$$2^5 = 2 \ast 2 \ast 2 \ast 2 \ast 2$$
$$2^5 \ast 2^3 = (2 \ast 2 \ast 2 \ast 2 \ast 2) \ast (2 \ast 2 \ast 2) = 2^8$$
$$\ (2^3)^5 = (2 \ast 2 \ast 2) \ast (2 \ast 2 \ast 2) \ast (2 \ast 2 \ast 2) \ast (2 \ast 2 \ast 2) \ast (2 \ast 2 \ast 2) = 2^{15}$$
$$2^{3^5} = 2^{243} \gg (2^3)^5 = 2^{15}$$
$$2^3 \cdot 5^3 = (2 \ast 2 \ast 2) \ast (5 \ast 5 \ast 5) = (2 \ast 5) \ast (2 \ast 5) \ast (2 \ast 5) = (2 \ast 5)^3$$
Pushing it further...

Take for granted: If $n$ and $m$ are positive integers,

$$a^n = a \cdot a \cdot \cdots \cdot a, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n \cdot m}.$$ 

Notice:

1. What is $a^0$?
   
   $a^n \cdot a^0 = a^n$, so $a^0 = 1$.

2. What is $a^n$ if $n$ is negative?
   
   $a^n \cdot a^m = a^{n+m} = 1$, so $a^n = 1/a^m$.

3. What is $a^n$ if $n$ is a fraction?
   
   $(a^n)^{1/n} = a^{n \cdot 1/n} = a^{n/n} = a^1$, so $a^{1/n} = \sqrt[n]{a}$ and $a^{m/n} = \sqrt[n]{a^m}$.
Pushing it further...

Take for granted: If $n$ and $m$ are positive integers,

\[
a^n = a \cdot a \cdots \cdot a, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n \cdot m}.
\]

Notice:

1. What is $a^0$?

\[
a^n = a^{n+0} = a^n \cdot a^0
\]

Example: $8^{5/3} = \sqrt[3]{8^5} = 2^5 = 32$ or $8^{5/3} = \sqrt[3]{8^{15}} = (8^5)^{1/3} = 32^{1/3} = 2$, 768 = 32
Pushing it further...

Take for granted: If $n$ and $m$ are positive integers,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_n, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n\cdot m}.$$

Notice:

1. What is $a^0$?

$$a^n = a^{n+0} = a^n \cdot a^0, \quad \text{so } a^0 = 1.$$
Pushing it further...

Take for granted: If $n$ and $m$ are positive integers,

$$a^n = a \cdot a \cdot \cdots \cdot a, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n\cdot m}.$$ 

Notice:

1. What is $a^0$?

   $$a^n = a^{n+0} = a^n \cdot a^0,$$
   
   so $[a^0 = 1]$. 

2. What is $a^x$ if $x$ is negative?

   $$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1$$
Pushing it further...

Take for granted: If $n$ and $m$ are positive integers,

$$a^n = a \cdot a \cdots a, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n\cdot m}.$$ 

Notice:

1. What is $a^0$?

   $$a^n = a^{n+0} = a^n \cdot a^0, \quad \text{so} \quad a^0 = 1.$$ 

2. What is $a^x$ if $x$ is negative?

   $$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1, \quad \text{so} \quad a^{-n} = 1/(a^n).$$
Pushing it further...

Take for granted: If \( n \) and \( m \) are positive integers,

\[
a^n = a \cdot a \cdot \cdots \cdot a, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n \cdot m}.
\]

Notice:

1. What is \( a^0 \)?
   \[
a^n = a^{n+0} = a^n \cdot a^0,
   \]
   so \( a^0 = 1 \).

2. What is \( a^x \) if \( x \) is negative?
   \[
a^n \cdot a^{-n} = a^{n-n} = a^0 = 1,
   \]
   so \( a^{-n} = 1/(a^n) \).

3. What is \( a^x \) if \( x \) is a fraction?
   \[
   (a^n)^{1/n} = a^{n \cdot \frac{1}{n}} = a^1 = a
   \]
Pushing it further...

Take for granted: If \( n \) and \( m \) are positive integers,

\[
a^n = a \cdot a \cdot \cdots \cdot a, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n \cdot m}.
\]

Notice:

1. What is \( a^0 \)?

\[
a^n = a^{n+0} = a^n \cdot a^0, \quad \text{so} \quad a^0 = 1.
\]

2. What is \( a^x \) if \( x \) is negative?

\[
a^n \cdot a^{-n} = a^{n-n} = a^0 = 1, \quad \text{so} \quad a^{-n} = 1/(a^n).
\]

3. What is \( a^x \) if \( x \) is a fraction?

\[
(a^n)^{\frac{1}{n}} = a^{n \cdot \frac{1}{n}} = a^1 = a, \quad \text{so} \quad a^{1/n} = \sqrt[n]{a}
\]

and \( a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \).
Pushing it further...

Take for granted: If $n$ and $m$ are positive integers,

$$a^n = a \cdot a \cdot \cdots \cdot a; \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n \cdot m}.$$ 

Notice:

1. What is $a^0$?

$$a^n = a^{n+0} = a^n \cdot a^0, \quad \text{so } a^0 = 1.$$

2. What is $a^x$ if $x$ is negative?

$$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1, \quad \text{so } a^{-n} = \frac{1}{a^n}.$$ 

3. What is $a^x$ if $x$ is a fraction?

$$\left(a^n\right)^{1/n} = a^{n\cdot \frac{1}{n}} = a^1 = a, \quad \text{so } a^{1/n} = \sqrt[n]{a}.$$ 

and $$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m.$$

Example: $8^{5/3} = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32$ or $8^{5/3} = \sqrt[3]{8^5} = \sqrt[3]{32,768} = 32$
What is $a^x$ for all $x$?

If $a > 1$:

(e.g. $a = 2$)

$x = 1, 2, 3, \ldots$
What is $a^x$ for all $x$?

If $a > 1$:

$\begin{array}{c|c|c|c|c|c|c|c|c|c|c|}
 x & \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \\
\hline
(e.g. \quad a = 2) & & & & & & & & & \\
\end{array}$
What is $a^x$ for all $x$?

If $a > 1$:

$$x = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$
What is $a^x$ for all $x$?

If $a > 1$:

$\overset{\text{e.g. } a = 2}{\text{(} x = n/2, \text{ for } n = 0, \pm 1, \pm 2, \pm 3, \ldots \text{)}}$
What is $a^x$ for all $x$?

If $a > 1$:

(e.g. $a = 2$)

$x = n/2$ and $n/3$, for $n = 0, \pm1, \pm2, \pm3, \ldots$
What is $a^x$ for all $x$?

If $a > 1$:

(e.g. $a = 2$)

$x = \frac{n}{2}, \frac{n}{3}, \ldots, \frac{n}{15}$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
What is $a^x$ for all $x$?

If $a > 1$:

(e.g. $a = 2$)

$x = n/2, n/3, \ldots, n/100$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
What is $a^x$ for all $x$?

If $a > 1$:

(e.g. $a = 2$)
What is $a^x$ for all $x$?

If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)

$x = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
What is $a^x$ for all $x$?

If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)

$x = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
What is $a^x$ for all $x$?

If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)

$x = \frac{n}{2}, \frac{n}{3}, \frac{n}{4}, \frac{n}{5}$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
What is $a^x$ for all $x$?

If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)

$x = \frac{n}{2}, \frac{n}{3}, \ldots, \frac{n}{100}$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
What is $a^x$ for all $x$?

If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)
What is $a^x$ for all $x$?

If $0 > a$:

$\begin{align*}
\text{e.g. } a &= -2 \\
\begin{array}{cccccc}
x &= \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots 
\end{array}
\end{align*}$
What is $a^x$ for all $x$?

If $0 > a$:

$$x = n/3, \text{ for } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$
What is $a^x$ for all $x$?

If $0 > a$:

$\begin{align*}
  x &= \frac{n}{3} \text{ and } \frac{n}{2}, \text{ for } n = 0, \pm 1, \pm 2, \pm 3, \ldots
\end{align*}$
What is $a^x$ for all $x$?

If $0 > a$:

(e.g. $a = -2$)

$x = n/2, n/3, \ldots, n/100$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

OH NO!
The function $a^x$:

$1 < a:$

D: $(-\infty, \infty)$, R: $(0, \infty)$

$0 < a < 1:$

D: $(-\infty, \infty)$, R: $(0, \infty)$

$a = 1:$

D: $(-\infty, \infty)$, R: $\{1\}$

$a = 0:$

D: $(0, \infty)$, R: $\{0\}$

Properties:

$a^b \cdot a^c = a^{b+c} \quad (a^b)^c = a^{b\cdot c} \quad a^{-x} = 1/a^x \quad a^c \cdot b^c = (ab)^c$
Our favorite exponential function:

Look at how the function is increasing through the point (0, 1):

\[ y = a^x : \]

\[ a = 1.1 \]
\[ a = 1.5 \]
\[ a = 2 \]
\[ a = 3 \]
\[ a = 10 \]

Q: Is there an exponential function whose slope at (0, 1) is 1?

A: \( e^x \) is the exponential function whose slope at (0, 1) is 1. (\( e \approx 2.71828183 \) is to calculus as \( \pi \approx 3.14159265 \) is to geometry)
Our favorite exponential function:

Look at how the function is increasing through the point \((0, 1)\):

\[ y = a^x : \]

\(a = 1.1\)
Our favorite exponential function:

Look at how the function is increasing through the point (0, 1):

\[ y = a^x : \]

\[ a=1.5 \]

Q: Is there an exponential function whose slope at (0,1) is 1?
A: \( e^x \) is the exponential function whose slope at (0,1) is 1.

(\( e \approx 2.71828183 \) is to calculus as \( \pi \approx 3.14159265 \) is to geometry)
Our favorite exponential function:

Look at how the function is increasing through the point \((0, 1)\):

\[ y = a^x : \]

\(a = 2\)
Our favorite exponential function:

Look at how the function is increasing through the point \((0, 1)\):

\[
y = a^x:
\]

\(a = 3\)

\(e^x\) is the exponential function whose slope at \((0, 1)\) is 1.

\(\pi = 3.14159265...\) is to calculus as \(\pi = 3.14159265...\) is to geometry.
Our favorite exponential function:

Look at how the function is increasing through the point (0, 1):

\[ y = a^x : \]

a = 10

\[ e^x \text{ is the exponential function whose slope at (0,1) is 1.} \]

\( e = 2.71828183 \) is to calculus as \( \pi = 3.14159265 \) is to geometry.
Our favorite exponential function:

Look at how the function is increasing through the point $(0, 1)$:

$$y = a^x :$$

Q: Is there an exponential function whose slope at $(0, 1)$ is 1?
Our favorite exponential function:

Look at how the function is increasing through the point \((0, 1)\):

\[
y = a^x:
\]

Q: Is there an exponential function whose slope at \((0,1)\) is 1?
Our favorite exponential function:

Look at how the function is increasing through the point (0, 1):

\[ y = a^x : \]

\[ a = 2.71828183... \]

Q: Is there an exponential function whose slope at (0,1) is 1?
Our favorite exponential function:

Look at how the function is increasing through the point \((0, 1)\):

\[ y = a^x : \]

\[ e = 2.71828183... \]

Q: Is there an exponential function whose slope at \((0, 1)\) is 1?
A: \(e^x\) is the exponential function whose slope at \((0, 1)\) is 1.
\((e = 2.71828183... \text{ is to calculus as } \pi = 3.14159265... \text{ is to geometry})\)
Logarithms

The exponential function $a^x$ has inverse $\log_a(x)$.
Logarithms

The exponential function $a^x$ has inverse $\log_a(x)$, i.e.

$$\log_a(a^x) = x = a^{\log_a(x)}$$
The exponential function $a^x$ has inverse $\log_a(x)$, i.e.

$$\log_a(a^x) = x = a^{\log_a(x)}, \quad \text{i.e.}$$

$$y = a^x \quad \text{if and only if} \quad \log_a(y) = x.$$
Logarithms

The exponential function \( a^x \) has inverse \( \log_a(x) \), i.e.

\[
\log_a(a^x) = x = a^{\log_a(x)}, \quad \text{i.e.}
\]

\[
y = a^x \quad \text{if and only if} \quad \log_a(y) = x.
\]
Properties of Logarithms

Domain: $(0, 1)$ i.e. all $x > 0$
Range: $(1, 1)$ i.e. all $x$
Properties of Logarithms

\[ y = \log_a(x) \]

Domain: \((0, 1)\) i.e. all \(x > 0\)
Range: \((1, \infty)\) i.e. all \(x > 0\)

\[ a = 10 \]
\[ a = 1.1 \]
Properties of Logarithms

Domain: \((0, \infty)\) i.e. all \(x > 0\)  
Range: \((-\infty, \infty)\) i.e. all \(x\)
Properties of Logarithms

0 < a < 1:

Domain: \((0, \infty)\) i.e. all \(x > 0\)

Range: \((-\infty, \infty)\) i.e. all \(x\)
Properties of Logarithms

Since... we know...
Properties of Logarithms

Since . . .

1. $a^0 = 1$

we know . . .

1. $\log_a(1) = 0$
Properties of Logarithms

Since...

1. \( a^0 = 1 \)
2. \( a^1 = a \)

we know...

1. \( \log_a(1) = 0 \)
2. \( \log_a(a) = 1 \)
Properties of Logarithms

Since...

1. \( a^0 = 1 \)
2. \( a^1 = a \)
3. \( a^b \times a^c = a^{b+c} \)

we know...

1. \( \log_a(1) = 0 \)
2. \( \log_a(a) = 1 \)
3. \( \log_a(b \times c) = \log_a(b) + \log_a(c) \)

Example:

Suppose \( y = \log_a(b^c) + \log_a(c) \). Then

\[
a^y = a^{\log_a(b^c)} \times a^{\log_a(c)} = b^c \times c.
\]

So \( y = \log_a(b^c) \) as well!
Properties of Logarithms

Since...

1. \( a^0 = 1 \)
2. \( a^1 = a \)
3. \( a^b \cdot a^c = a^{b+c} \)

we know...

1. \( \log_a(1) = 0 \)
2. \( \log_a(a) = 1 \)
3. \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \)

Example: why \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \):
Suppose \( y = \log_a(b) + \log_a(c) \).
Properties of Logarithms

Since...

1. $a^0 = 1$
2. $a^1 = a$
3. $a^b \times a^c = a^{b+c}$

we know...

1. $\log_a(1) = 0$
2. $\log_a(a) = 1$
3. $\log_a(b \times c) = \log_a(b) + \log_a(c)$

Example: why $\log_a(b \times c) = \log_a(b) + \log_a(c)$:
Suppose $y = \log_a(b) + \log_a(c)$.

Then $a^y = a^{\log_a(b) + \log_a(c)}$
Properties of Logarithms

Since...

1. $a^0 = 1$
2. $a^1 = a$
3. $a^b \cdot a^c = a^{b+c}$

we know...

1. $\log_a(1) = 0$
2. $\log_a(a) = 1$
3. $\log_a(b \cdot c) = \log_a(b) + \log_a(c)$

Example: why $\log_a(b \cdot c) = \log_a(b) + \log_a(c)$:

Suppose $y = \log_a(b) + \log_a(c)$.

Then $a^y = a^{\log_a(b)+\log_a(c)} = a^{\log_a(b)} a^{\log_a(c)}$
Properties of Logarithms

Since...

1. $a^0 = 1$
2. $a^1 = a$
3. $a^b \times a^c = a^{b+c}$

we know...

1. $\log_a(1) = 0$
2. $\log_a(a) = 1$
3. $\log_a(b \times c) = \log_a(b) + \log_a(c)$

Example: why $\log_a(b \times c) = \log_a(b) + \log_a(c)$:
Suppose $y = \log_a(b) + \log_a(c)$.

Then $a^y = a^{\log_a(b) + \log_a(c)} = a^{\log_a(b)}a^{\log_a(c)} = b \times c$. 

Properties of Logarithms

Since... we know...

1. \( a^0 = 1 \)  
2. \( a^1 = a \)  
3. \( a^b \cdot a^c = a^{b+c} \)

1. \( \log_a(1) = 0 \)  
2. \( \log_a(a) = 1 \)  
3. \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \)

Example: why \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \):
Suppose \( y = \log_a(b) + \log_a(c) \).

Then \( a^y = a^{\log_a(b) + \log_a(c)} = a^{\log_a(b)} a^{\log_a(c)} = b \cdot c \).

So \( y = \log_a(b \cdot c) \) as well!
Properties of Logarithms

Since...

1. \( a^0 = 1 \)
2. \( a^1 = a \)
3. \( a^b \times a^c = a^{b+c} \)
4. \( (a^b)^c = a^{b \times c} \)

we know...

1. \( \log_a(1) = 0 \)
2. \( \log_a(a) = 1 \)
3. \( \log_a(b \times c) = \log_a(b) + \log_a(c) \)
4. \( \log_a(b^c) = c \log_a(b) \)

Example: why \( \log_a(b \times c) = \log_a(b) + \log_a(c) \):

Suppose \( y = \log_a(b) + \log_a(c) \).

Then \( a^y = a^{\log_a(b)+\log_a(c)} = a^{\log_a(b)} a^{\log_a(c)} = b \times c \).

So \( y = \log_a(b \times c) \) as well!
Properties of Logarithms

Since...

1. \( a^0 = 1 \)
2. \( a^1 = a \)
3. \( a^b \cdot a^c = a^{b+c} \)
4. \( (a^b)^c = a^{b\cdot c} \)

we know...

1. \( \log_a(1) = 0 \)
2. \( \log_a(a) = 1 \)
3. \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \)
4. \( \log_a(b^c) = c \log_a(b) \)

Example: why \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \):
Suppose \( y = \log_a(b) + \log_a(c) \).

Then \( a^y = a^{\log_a(b) + \log_a(c)} = a^{\log_a(b)} a^{\log_a(c)} = b \cdot c \).

So \( y = \log_a(b \cdot c) \) as well!

Lastly: \( \frac{\log_a(b)}{\log_a(c)} = \log_c(b) \)
4. Let \( y = c \log_a(b) \).

Then \( a^y = a^{c \log_a(b)} \)

\[ = (a^{\log_a(c)})^{\log_c(b)} \]

\[ = b^c. \]

So \( y = \log_c(b^c). \)

-- or --

Let \( y = \log_a(b^c). \)

Then \( a^y = b^c \)

\[ = (a^{\log_a(b)})^c \]

\[ = a^{c \log_a(b)} \]

\[ = a. \]

So \( y = \log_a(b). \)

---

Change of base:

Let \( y = \log_a(c) \cdot \log_d(b) \)

Then \( a^y = a^{(\log_a(c) \cdot \log_d(b))} \)

\[ = (a^{\log_a(c)})^{\log_d(b)} \]

\[ = b^{\log_d(c)}. \]

So \( y = \log_d(c). \)

Thus \( \log_a(b) = \log_a(c) \cdot \log_c(b) \)

So \( \frac{\log_a(b)}{\log_a(c)} = \log_c(b). \)
Favorite logarithmic function

Remember: \( y = e^x \) is the function whose slope through the point (0,1) is 1.
The inverse to \( y = e^x \) is the *natural log*:

\[
\ln(x) = \log_e(x)
\]
Favorite logarithmic function

Remember: $y = e^x$ is the function whose slope through the point $(0,1)$ is 1. The inverse to $y = e^x$ is the natural log:

$$\ln(x) = \log_e(x)$$

We will often use the facts that $e^{\ln(x)} = x$ (for $x > 0$) and $\ln(e^x) = x$ (for all $x$)
Two super useful facts:

Explain why:

(1) $\log_a(b) = \ln(b)/\ln(a)$

(2) $a^b = e^{b \ln(a)}$ [hint: start by rewriting $b \ln(a)$, and use the fact that $e^{\ln(x)} = x$]
Two super useful facts:

Explain why:

(1) $\log_a(b) = \ln(b) / \ln(a)$

Since $\ln(b) = \log_e(b)$ and $\ln(a) = \log_e(a)$, we have

$$\frac{\ln(b)}{\ln(a)} = \frac{\log_e(b)}{\log_e(a)} = \log_a(b)$$

(2) $a^b = e^{b \ln(a)}$ [hint: start by rewriting $b \ln(a)$, and use the fact that $e^{\ln(x)} = x$]

Since $b \ln(a) = \ln(a^b)$ and $e^{\ln(x)} = x$, we have

$$e^{b \ln(a)} = e^{\ln(a^b)} = a^b$$
Examples:

(1) Condense the logarithmic expressions

\[
\frac{1}{2} \ln(x) + 3 \ln(x+1) \quad 2 \ln(x+5) - \ln(x) \quad \frac{1}{3} (\log_3(x) - \log_3(x+1))
\]

(2) Solve the following expressions for \( x \):

\[
e^{-x^2} = e^{-3x-4} \quad 3(2^x) = 24
\]

\[
2(e^{3x-5}) - 5 = 11 \quad \ln(3x + 1) - \ln(5) = \ln(2x)
\]
Examples:

(1) Condense the logarithmic expressions

\[
\frac{1}{2} \ln(x) + 3 \ln(x+1) \quad 2 \ln(x+5) - \ln(x) \quad \frac{1}{3} \left( \log_3(x) - \log_3(x+1) \right)
\]

\[
\ln(\sqrt{x}(x+1)^3) \quad \ln \left( \frac{(x+5)^2}{x} \right) \quad \log_3 \left( \left( \frac{x}{x+1} \right)^{1/3} \right)
\]

(2) Solve the following expressions for \(x\):

\[
e^{-x^2} = e^{-3x-4} \quad 3(2^x) = 24
\]

\[
x = -1, 4 \quad x = 3
\]

\[
2(e^{3x-5}) - 5 = 11 \quad \ln(3x + 1) - \ln(5) = \ln(2x)
\]

\[
x = \frac{\ln(8)+5}{3} \quad x = \frac{1}{7}
\]
\[
\frac{1}{2} \ln(x) + 3 \ln(x+1) = \ln(x^{\frac{1}{2}}) + \ln((x+1)^3)
\]
\[
= \ln\left(x^{\frac{1}{2}}(x+1)^3\right)
\]
\[
2 \ln(x+5) - \ln(x) = \ln((x+5)^2) + \ln(x^{-1})
\]
\[
= \ln\left((x+5)^2 \cdot x^{-1}\right) = \ln\left(\frac{(x+5)^2}{x}\right)
\]
\[
\frac{1}{3} \left(\log_3(x) - \log_3(x+1)\right) = \frac{1}{3} \left(\log_3\left(x\cdot(x+1)^{-1}\right)\right)
\]
\[
= \log_3\left(\frac{3}{\sqrt[3]{x}}\right)
\]

\[x = 4, 1\]  \[x = 3\]
If \( e^{-x^2} = e^{-3x-4} \), (take \( \ln(\cdot) \) both sides)

then \(-x^2 = -3x - 4\),

so \( x^2 - 3x - 4 = 0 \)

\( (x-4)(x+1) = 0 \)

so \( x = 4 \) or \( -1 \)

If \( 3(2^x) = 24 \), then

\( 2^x = 8 \),

so \( x = 3 \)
If \(2(e^{3x-5})-5 = 11\),

then \(e^{3x-5} = \frac{11+5}{2} = 8\)

so \(3x-5 = \ln(8)\)

so \(x = \frac{\ln(8)+5}{3}\)
If \( \ln(3x+1) - \ln(5) = \ln(2x) \)

\[
\left( \ln \left( \frac{3x+1}{5} \right) \right) = \left( \ln(2x) \right)
\]

\[
\frac{3x+1}{5} = 2x
\]

\[
3x + 1 = 10x
\]

\[
x = \frac{1}{7}
\]