Exponential and Logarithmic Functions

The Basics
If \( n \) and \( m \) are positive integers...

\[ a^n = a \cdot a \cdot \ldots \cdot a \quad \text{(WeBWoRK: } a^n \text{ or } a \ast n) \]

Some identities:
\[ a^n \cdot a^m = a^{n+m} \quad (a^n)^m = a^{n \cdot m} \]
(Notice: \( a^m \) means \( a^{(m)} \), since \( (a^n)^n \) can be written another way)

Examples:
\[ a^n \cdot b^n = (a \cdot b)^n \]

\[ 2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \]
\[ 2^5 \cdot 2^3 = (2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^8 \]
\[ (2^3)^5 = (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) \cdot (2 \cdot 2 \cdot 2) = 2^{15} \]
\[ 2^{35} = 2^{243} \gg (2^3)^5 = 2^{15} \]
\[ 2^3 \cdot 5^3 = (2 \cdot 2 \cdot 2) \cdot (5 \cdot 5 \cdot 5) = (2 \cdot 5) \cdot (2 \cdot 5) \cdot (2 \cdot 5) = (2 \cdot 5)^3 \]

Pushing it further...
Take for granted: If \( n \) and \( m \) are positive integers,

\[ a^n = a \cdot a \cdot \ldots \cdot a, \quad a^n \cdot a^m = a^{n+m}, \quad (a^n)^m = a^{n^m}. \]

Notice:
1. What is \( a^0 \)?
\[ a^n = a^{n+0} = a^n \cdot a^0, \quad \text{so } a^0 = 1. \]
2. What is \( a^x \) if \( x \) is negative?
\[ a^n \cdot a^{-n} = a^{n-n} = a^0 = 1, \quad \text{so } a^{-n} = 1/(a^n). \]
3. What is \( a^x \) if \( x \) is a fraction?
\[ (a^n)^{1/n} = a^{n \cdot \frac{1}{n}} = a^1 = a, \quad \text{so } a^{1/n} = \sqrt[n]{a} \]
\[ \text{and } a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m. \]

Example: \( 8^{5/3} = (\sqrt[3]{8})^5 = 2^5 = 32 \) or \( 8^{5/3} = \sqrt[3]{8^5} = \sqrt[3]{32,768} = 32 \)
What is $a^x$ for all $x$?

If $a > 1$:

(e.g. $a = 2$)

$x = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

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$x = n/2$ and $n/3$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
What is $a^x$ for all $x$?
If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)

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If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)

$x = \frac{n}{2}, \frac{n}{3}, \ldots, \frac{n}{100}$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

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If $0 < a < 1$:

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What is $a^x$ for all $x$?
If $0 < a < 1$:

(e.g. $a = \frac{1}{2}$)

$y = a^x$
What is $a^x$ for all $x$?

If $0 > a$:

- For $a = -2$:
  - $x = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
  - $x = n/3$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

- For $a = -2$:
  - $x = n/3$ and $n/2$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
  - $x = n/2, n/3, \ldots, n/100$, for $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ OH NO!
The function $a^x$:

![Graphs of $a^x$ for different values of $a$]

Properties:

- $a^b \cdot a^c = a^{b+c}$
- $(a^b)^c = a^{b\cdot c}$
- $a^{-x} = \frac{1}{a^x}$
- $a^c \cdot b^c = (ab)^c$

Our favorite exponential function:

Look at how the function is increasing through the point $(0,1)$:
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Look at how the function is increasing through the point (0, 1):

\[ y = a^x : \]

a = 1.1

Q: Is there an exponential function whose slope at (0, 1) is 1?
A: \( e^x \) is the exponential function whose slope at (0, 1) is 1.

(\( e = 2.71828183 \ldots \) is to calculus as \( \pi = 3.14159265 \ldots \) is to geometry)
Our favorite exponential function:

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Logarithms

The exponential function $a^x$ has inverse $\log_a(x)$, i.e.

$$\log_a(a^x) = x = a^{\log_a(x)}, \text{ i.e.}$$

$$y = a^x \text{ if and only if } \log_a(y) = x.$$

Properties of Logarithms

Domain: $(0, \infty)$ i.e. all $x > 0$ \hspace{1cm} Range: $(-\infty, \infty)$ i.e. all $x$
Properties of Logarithms

Since... we know...

1. \( a^0 = 1 \)
2. \( a^1 = a \)
3. \( a^b \cdot a^c = a^{b+c} \)
4. \( (a^b)^c = a^{b+c} \)
1. \( \log_a(1) = 0 \)
2. \( \log_a(a) = 1 \)
3. \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \)
4. \( \log_a(b^c) = c \log_a(b) \)

Example: why \( \log_a(b \cdot c) = \log_a(b) + \log_a(c) \):

Suppose \( y = \log_a(b) + \log_a(c) \).

Then \( a^y = a^{\log_a(b) + \log_a(c)} = a^{\log_a(b)}a^{\log_a(c)} = b \cdot c \).

So \( y = \log_a(b \cdot c) \) as well!

Lastly: \( \frac{\log_a(b)}{\log_a(c)} = \log_c(b) \)

Favorite logarithmic function

Remember: \( y = e^x \) is the function whose slope through the point (0,1) is 1.
The inverse to \( y = e^x \) is the natural log:

\[ \ln(x) = \log_e(x) \]

We will often use the facts that \( e^{\ln(x)} = x \) (for \( x > 0 \)) and \( \ln(e^x) = x \) (for all \( x \))
Two super useful facts:

Explain why:

(1) $\log_a(b) = \ln(b)/\ln(a)$

(2) $a^b = e^{b \ln(a)}$ [hint: start by rewriting $b \ln(a)$, and use the fact that $e^{\ln(x)} = x$]

Examples:

(1) Condense the logarithmic expressions

\[
\frac{1}{2} \ln(x) + 3 \ln(x+1) \quad 2 \ln(x+5) - \ln(x) \quad \frac{1}{3} (\log_3(x) - \log_3(x+1))
\]

(2) Solve the following expressions for $x$:

\[
e^{-x^2} = e^{-3x-4} \quad 3(2^x) = 24
\]

\[
2(e^{3x-5}) - 5 = 11 \quad \ln(3x + 1) - \ln(5) = \ln(2x)
\]