Using implicit differentiation for good: Inverse functions.

Warmup: Calculate $\frac{dy}{dx}$ if
1. $e^y = xy$

2. $\cos(y) = x + y$

Every time:
(1) Take $\frac{dy}{dx}$ of both sides.
(2) Add and subtract to get the $\frac{dy}{dx}$ terms on one side and everything else on the other.
(3) Factor out $\frac{dy}{dx}$ and divide both sides by its coefficient.

The Derivative of $y = \ln x$

Remember:
(1) $y = e^x$ has a slope through the point (0,1) of 1.
(2) The natural log is the inverse to $e^x$, so

$$ y = \ln x \implies e^y = x $$

![Graph showing the relationship between $e^x$ and $\ln x$.](image)
The Derivative of \( y = \ln x \)

To find the derivative of \( \ln(x) \), use implicit differentiation!

Rewrite

\[ y = \ln x \quad \text{as} \quad e^y = x \]

Take a derivative of both sides of \( e^y = x \) to get

\[ \frac{dy}{dx} e^y = 1 \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{e^y} \]

**Problem:** We asked “what is the derivative of \( \ln(x) \)?” and got back and answer with \( y \) in it!

**Solution:** Substitute back!

\[ \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{e^{\ln(x)}} = \frac{1}{x} \]

\[ \frac{d}{dx} \ln(x) = \frac{1}{x} \]

Does it make sense?
Examples

Calculate
1. \( \frac{d}{dx} \ln x^2 \)
2. \( \frac{d}{dx} \ln(\sin(x^2)) \)
3. \( \frac{d}{dx} \log_3(x) \)
   
   [hint: \( \log_a x = \frac{\ln x}{\ln a} \)]

Quick tip: Logarithmic differentiation

Example: Calculate \( \frac{dy}{dx} \) if \( y = x^{\sin(x)} \)

Problem: Both the base and the exponent have the variable in them! So we can’t use

\[
\frac{d}{dx} x^a = ax^{a-1} \quad \text{or} \quad \frac{d}{dx} a^x = \ln(a) a^x.
\]

Fix: Take the log of both sides and use implicit differentiation:

\[
\ln(y) = \ln(x^{\sin(x)}) = \sin(x) \ast \ln(x) \quad \text{(using } \ln(a^b) = b \ln(a))
\]

Taking the derivative of both sides gives

\[
\frac{1}{y} \frac{dy}{dx} = \cos(x) \ln(x) + \sin(x) \frac{1}{x}
\]

Then solving for \( \frac{dy}{dx} \),

\[
\frac{dy}{dx} = y \left( \cos(x) \ln(x) + \sin(x) \frac{1}{x} \right) = x^{\sin(x)} \left( \cos(x) \ln(x) + \sin(x) \frac{1}{x} \right).
\]
Back to inverses

In the case where \( y = \ln(x) \), we used the fact that \( \ln(x) = f^{-1}(x) \), where \( f(x) = e^x \), and got

\[
\frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}}.
\]

In general, calculating \( \frac{d}{dx} f^{-1}(x) \):

1. Rewrite \( y = f^{-1}(x) \) as \( f(y) = x \).
2. Use implicit differentiation:

\[
f'(y) \cdot \frac{dy}{dx} = 1 \quad \text{so} \quad \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.
\]

Examples

Just to check, use the rule

\[
\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}
\]

to calculate

1. \( \frac{d}{dx} \ln(x) \) (the inverse of \( e^x \))
   In the notation above, \( f^{-1}(x) = \ln(x) \) and \( f(x) = e^x \).
   We’ll also need \( f'(x) = e^x \). So
   \[
   \frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}} \quad \odot
   \]
2. \( \frac{d}{dx} \sqrt{x} \) (the inverse of \( x^2 \))
   In the notation above, \( f^{-1}(x) = \sqrt{x} \) and \( f(x) = x^2 \).
   We’ll also need \( f'(x) = 2x \). So
   \[
   \frac{d}{dx} \sqrt{x} = \frac{1}{2 \cdot (\sqrt{x})} \quad \odot
   \]
Inverse trig functions

Two notations:

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$f^{-1}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(x)$</td>
<td>$\sin^{-1}(x) = \arcsin(x)$</td>
</tr>
<tr>
<td>$\cos(x)$</td>
<td>$\cos^{-1}(x) = \arccos(x)$</td>
</tr>
<tr>
<td>$\tan(x)$</td>
<td>$\tan^{-1}(x) = \arctan(x)$</td>
</tr>
<tr>
<td>$\sec(x)$</td>
<td>$\sec^{-1}(x) = \text{arcsec}(x)$</td>
</tr>
<tr>
<td>$\csc(x)$</td>
<td>$\csc^{-1}(x) = \text{arccsc}(x)$</td>
</tr>
<tr>
<td>$\cot(x)$</td>
<td>$\cot^{-1}(x) = \text{arccot}(x)$</td>
</tr>
</tbody>
</table>

There are lots of points we know on these functions...

Examples:

1. Since $\sin(\pi/2) = 1$, we have $\arcsin(1) = \pi/2$
2. Since $\cos(\pi/2) = 0$, we have $\arccos(0) = \pi/2$

Etc...

In general:

$\text{arc}_\_\_\_ (\_\_\_\_ \_\_\_\_\_\_)$ takes in a ratio and spits out an angle:

\[
\begin{aligned}
c &> b \\
\theta &> 0
\end{aligned}
\]

\[
\begin{aligned}
\cos(\theta) &= a/c \\
\sin(\theta) &= b/c \\
\tan(\theta) &= b/a
\end{aligned}
\]

so

\[
\begin{aligned}
\arccos(a/c) &= \theta \\
\arcsin(b/c) &= \theta \\
\arctan(b/a) &= \theta
\end{aligned}
\]

Domain problems:

$\sin(0) = 0, \quad \sin(\pi) = 0, \quad \sin(2\pi) = 0, \quad \sin(3\pi) = 0, \ldots$

So which is the right answer to $\arcsin(0)$, really?
\[ y = \sin(x) \]  
\[ y = \arcsin(x) \]

**Domain:** 
\[-1 \leq x \leq 1\]

**Range:** 
\[-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\]
\[ y = \cos(x) \]
\[ y = \arccos(x) \]

**Domain:** \[-1 \leq x \leq 1\]

**Range:** \[0 \leq y \leq \pi\]
$y = \tan(x)$

Domain: $1 \leq x \leq 1$

Range: $\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \arctan(x)$

Domain: $-\infty \leq x \leq \infty$

$y = \arctan(x)$

Domain: $-\infty \leq x \leq \infty$

Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$
\[ y = \sec(x) \]
\[ y = \text{arcsec}(x) \]

**Domain:** \(-1 \leq x \leq 1\) and \(x \neq 0\)

**Range:** \(0 \leq y \leq \pi\)
$$y = \csc(x)$$

Domain: $$x \leq -1 \text{ and } 1 \leq x$$

Range: $$\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$
\[ y = \cot(x) \]

\[ y = \arccot(x) \]

Domain: \(-\infty \leq x \leq \infty\)

Range: \(-\infty \leq x \leq \infty\)
Back to Derivatives

Use implicit differentiation to calculate the derivatives of

1. \( \arcsin(x) \)
2. \( \arctan(x) \)

Use the rule

\[
\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}
\]

to check your answers, and then to calculate the derivatives of the other inverse trig functions:

1. \( \frac{d}{dx} \arccos(x) \)
2. \( \frac{d}{dx} \arcsec(x) \)
3. \( \frac{d}{dx} \arccsc(x) \)
4. \( \frac{d}{dx} \arccot(x) \)

Recall:

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( f'(x) )</th>
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<tbody>
<tr>
<td>( \sin(x) )</td>
<td>( \cos(x) )</td>
</tr>
<tr>
<td>( \cos(x) )</td>
<td>( -\sin(x) )</td>
</tr>
<tr>
<td>( \tan(x) )</td>
<td>( \sec^2(x) )</td>
</tr>
<tr>
<td>( \sec(x) )</td>
<td>( \sec(x)\tan(x) )</td>
</tr>
<tr>
<td>( \csc(x) )</td>
<td>( -\csc(x)\cot(x) )</td>
</tr>
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<td>( \cot(x) )</td>
<td>( -\csc^2(x) )</td>
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</tbody>
</table>
Using implicit differentiation to calculate $\frac{d}{dx} \arcsin(x)$

If $y = \arcsin(x)$ then $x = \sin(y)$.

Take $\frac{d}{dx}$ of both sides of $x = \sin(y)$:

Left hand side: $\frac{d}{dx} x = 1$

Right hand side: $\frac{d}{dx} \sin(y) = \cos(y) \frac{dy}{dx} = \cos(\arcsin(x)) \frac{dy}{dx}$

So

$$\frac{dy}{dx} = \frac{1}{\cos(\arcsin(x))}.$$

Simplifying $\cos(\arcsin(x))$

Call $\arcsin(x) = \theta$.

$\sin(\theta) = x$

Key: This is a simple triangle to write down whose angle $\theta$ has $\sin(\theta) = x$
Simplifying $\cos(\arcsin(x))$

Call $\arcsin(x) = \theta$.

$$\sin(\theta) = x$$

So $\cos(\arcsin(x)) = \sqrt{1 - x^2}$

So $\frac{d}{dx} \arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - x^2}}$.

Calculating $\frac{d}{dx} \arctan(x)$.

We found that

$$\frac{d}{dx} \arctan(x) = \frac{1}{\sec^2(\arctan(x))} = \left(\frac{1}{\sec(\arctan(x))}\right)^2$$

Simplify this expression using
To simplify the rest, use the triangles

\[
\begin{align*}
\text{arccos}(x) & \quad \text{arcsec}(x) \\
1 & \quad 1 \\
x & \quad 1 \\
x & \quad 1 \\
\text{arccsc}(x) & \quad \text{arccot}(x)
\end{align*}
\]