Going between graphs of functions and their derivatives:

Mean value theorem, Rolle’s theorem, and intervals of increase and decrease

Recall: The Intermediate Value Theorem

Suppose $f$ is continuous on a closed interval $[a, b]$.

If $f(a) < C < f(b)$ or $f(a) > C > f(b)$,

then there is at least one point $c$ in the interval $[a, b]$ such that

$$f(c) = C.$$
The Mean Value Theorem

**Theorem**
Suppose that $f$ is defined and continuous on a closed interval $[a, b]$, and suppose that $f'$ exists on the open interval $(a, b)$. Then there exists a point $c$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

**Bad examples**

- Discontinuity at an endpoint
- Discontinuity at an interior point
- No derivative at an interior point
Examples

Does the mean value theorem apply to \( f(x) = |x| \) on \([-1, 1]\)?

How about to \( f(x) = |x| \) on \([1, 5]\)?

Example

Under what circumstances does the Mean Value Theorem apply to the function \( f(x) = 1/x \)?
Verify the conclusion of the Mean Value Theorem for the function \( f(x) = (x + 1)^3 - 1 \) on the interval \([-3, 1]\).

**Step 1:** Check that the conditions of the MVT are met.

**Step 2:** Calculate the slope \( m \) of the line joining the two endpoints.

**Step 3:** Solve the equation \( f'(x) = m \).
Intervals on increase/decrease

Formally,

<table>
<thead>
<tr>
<th></th>
<th>( f(x + h) - f(x) )</th>
<th>( \lim_{h \to 0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f ) is increasing if</td>
<td>pos.</td>
<td>pos. or 0 (non-neg)</td>
</tr>
<tr>
<td>( f(x_1) &lt; f(x_2) ) whenever ( x_1 &lt; x_2 ).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f ) is nondecreasing if</td>
<td>non-neg.</td>
<td>non-neg.</td>
</tr>
<tr>
<td>( f(x_1) \leq f(x_2) ) whenever ( x_1 &lt; x_2 ).</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f ) is decreasing if</td>
<td>neg.</td>
<td>non-pos.</td>
</tr>
<tr>
<td>( f(x_1) &gt; f(x_2) ) whenever ( x_1 &lt; x_2 ).</td>
<td></td>
<td></td>
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<tr>
<td>( f ) is nonincreasing if</td>
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</table>

So we can calculate some of the “shape” of \( f(x) \) by knowing when its derivative is positive, negative, and 0!

Sign of the derivative

If \( f(x) \) is increasing, what is the sign of the derivative?

Look at the difference quotient:

\[
\frac{f(x + h) - f(x)}{h}
\]

The derivative is a two-sided limit, so we have two cases:

**Case 1:** \( h \) is positive.

So \( x + h > x \), which implies \( f(x + h) - f(x) > 0 \).

So

\[
\frac{f(x + h) - f(x)}{h} > 0.
\]

**Case 2:** \( h \) is negative.

So \( x + h < x \), which implies \( f(x + h) - f(x) < 0 \).

So

\[
\frac{f(x + h) - f(x)}{h} > 0.
\]

So the difference quotient is positive!
Example

On what interval(s) is the function $f(x) = x^3 + x + 1$ increasing or decreasing?

**Step 1:** Calculate the derivative.

**Step 2:** Decide when the derivative is positive, negative, or zero.

**Step 3:** Bring that information back to $f(x)$. 
Example

Find the intervals on which the function 
\( f(x) = 2x^3 - 6x^2 - 18x + 1 \) is increasing and those on which it is decreasing.

**Step 1:** Calculate the derivative.

**Step 2:** Decide when the derivative is positive, negative, or zero.

**Step 3:** Bring that information back to \( f(x) \).
If $f$ is continuous on a closed interval $[a, b]$, then there is a point in the interval where $f$ is largest (maximized) and a point where $f$ is smallest (minimized).

The maxima or minima will happen either
1. at an endpoint, or
2. at a **critical point**, a point $c$ where $f'(c) = 0$

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**Example**

For the function $f(x) = 2x^3 - 6x^2 - 18x + 1$, let us find the points in the interval $[-4, 4]$ where the function assumes its maximum and minimum values.

$$f'(x) = 6x^2 - 12x - 18 = 6(x - 3)(x + 1)$$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>-53</td>
</tr>
<tr>
<td>-4</td>
<td>-151</td>
</tr>
<tr>
<td>4</td>
<td>-39</td>
</tr>
</tbody>
</table>

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Rolle’s Theorem

**Theorem**
Suppose that the function $f$ is
continuous on the closed interval $[a, b]$,
differentiable on the open interval $(a, b)$, and
$a$ and $b$ are both roots of $f$.
Then there is at least one point $c$ in $(a, b)$ where $f'(c) = 0$.

(In other words, if $g$ didn’t jump, then it had to turn around)

Back to Newton’s method

Remember: Newton’s method helped us find roots of functions.

Pick an $x_0$ to start. To get $x_{i+1}$, follow the tangent line to $f(x)$ at $x_i$
down to its $x$-intercept. The $x_i$’s get closer and closer to a root of $f$.

But how do we know when we’ve found all of them?
For example: Find the roots of $f(x) = x^5 - 3x + 1$.

<table>
<thead>
<tr>
<th>If $x_0$ is . . .</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>then the $x_i$’s get closer to . . .</td>
<td>-1.3888</td>
<td>-1.3888</td>
<td>0.3347</td>
<td>1.2146</td>
<td>1.2146</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>$x_0$</th>
<th>-.9</th>
<th>-1.8</th>
<th>-.7</th>
<th>-.6</th>
<th>.5</th>
<th>.6</th>
<th>.7</th>
</tr>
</thead>
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<tr>
<td>$x_i$</td>
<td>-1.3 . .</td>
<td>1.2 . .</td>
<td>1.2 . .</td>
<td>0.3 . .</td>
<td>0.3 . .</td>
<td>0.3 . .</td>
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<table>
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<tr>
<th>$x_0$</th>
<th>-10</th>
<th>-20</th>
<th>-50</th>
<th>-100</th>
<th>-1000</th>
<th>-10000</th>
</tr>
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<td>$x_i$</td>
<td>-1.3 . .</td>
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</table>

After plugging in lots of $x_0$’s, we’ve only found three roots. But there
could be up to 5! How do we know we’re not just very unlucky?
Use Rolle’s Theorem to show that $f(x) = x^5 - 3x + 1$ has exactly three real roots!

**Step 1:** Show that there are at most three roots.

**Step 2:** Show that there are at least three roots.

Two methods:
1. Use Newton’s method to root out three roots, or
2. find four points $f(x)$ which alternate signs, and use the intermediate value theorem.

(IVT: If $f(x)$ is cont. and $f(a) < C < f(b)$, then there’s a $c$ btwn. $a$ and $b$ where $f(c) = C$)

On your own:

1. **Do an analysis of increasing/decreasing on** $f(x)$.
   How many times does $f(x)$ turn around?
   Conclude: what is an upper bound on the number of roots?

2. **Find the heights of the critical points.**
   Using the intermediate value theorem, what is a lower bound on the number of roots? Can you do better if you also find the height of the function at a big positive number and a big negative number?

3. **Conclude:** How many real roots does $f(x)$ have?

4. **Bonus:**
   Using the approximations from before, sketch a graph.