Warm-up

1. Calculate the area between the $x$-axis and the curve $y = -x^2 + 5x - 6$ between $x = 1$ and $x = 2$.
   (Your answer should be positive — we want area.)

2. Calculate the area of the region enclosed between the curve $y = -x^2 + 5x - 6$ and the $x$-axis.
   (Find where $y = -x^2 + 5x - 6$ intersects the $x$-axis to get bounds.)

3. Calculate the area contained between the curve $y = x^2 - 5x + 6$ and the $x$-axis.
   (Again, your answer should be positive.)

Tip: Sketch $y = -(x^2 - 5x + 6)$ before you do any of these problems)
Areas Between Curves

We know that if \( f \) is a continuous nonnegative function on the interval \([a, b]\), then \( \int_a^b f(x) \, dx \) is the area under the graph of \( f \) and above the interval.

Now suppose we are given two continuous functions, \( f(x) \) and \( g(x) \) so that \( g(x) \leq f(x) \) for all \( x \) in the interval \([a, b]\).

How do we find the area bounded by the two functions over that interval?

Area between \( f \) and \( g = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx = \int_a^b f(x) - g(x) \, dx \)
Area between \( f \) and \( g \) = \( \int_{a}^{b} f(x)dx - \int_{a}^{b} g(x)dx = \int_{a}^{b} f(x) - g(x)dx \)
Example

Find the area of the region between the graphs of \( y = x^2 \) and \( y = x^3 \) for \( 0 \leq x \leq 1 \).

\[
\text{Top: } x^2 \quad \text{Bottom: } x^3
\]
Intersections: where does \( x^2 = x^3 \)? \( x = 0 \) or \( 1 \)

So

\[
\text{Area} = \int_0^1 x^2 - x^3 \, dx = \left[ \frac{1}{3} x^3 - \frac{1}{4} x^4 \right]_0^1 = \left( \frac{1}{3} - \frac{1}{4} \right) - 0 > 0
\]

Example

Find the area of the region bounded by the two curves \( y = x^3 - 9x \) and \( y = 9 - x^2 \).

1. Check for intersection points (Solve \( x^3 - 9x = 9 - x^2 \)).

2. Area = Area A + Area B

\[
\text{Area A} = \int_{-3}^{-1} (x^3 - 9x) - (9 - x^2) \, dx = \int_{-3}^{-1} x^3 + x^2 - 9x - 9 \, dx
\]

\[
\text{Area B} = \int_{-1}^{3} (9 - x^2) - (x^3 - 9x) \, dx = -\int_{-1}^{3} x^3 + x^2 - 9x - 9 \, dx
\]
Functions of $y$

We could just as well consider two functions of $y$, say, $x = f_{\text{Left}}(y)$ and $x = g_{\text{Right}}(y)$ defined on the interval $[c, d]$.

Area Between the Two Curves

Find the area under the graph of $y = \ln x$ and above the interval $[1, e]$ on the $x$-axis.

$$
\text{area} = \int_{y=0}^{1} e - e^y \, dy = (e \cdot y - e^y)\big|_{y=0}^{1} = (e - e) - (0 - 1) = 1.
$$
Quick note: Putting FTC and substitution together

**Q.** Calculate \( \int_{0}^{\pi/2} x \sin(x^2) \, dx \).

**A.** Separate your solution into two steps.

**Step 1:** Find the antiderivative \( F(x) \) of \( f(x) = x \sin(x^2) \).

Let \( u = x^2 \). So \( du = 2x \, dx \), and \( \frac{1}{2} \, du = x \, dx \).

Therefore

\[
\int x \sin(x^2) \, dx = \int \sin(u) \cdot \frac{1}{2} \, du
= -\frac{1}{2} \cos(u) + C = -\frac{1}{2} \cos(x^2) + C
\]

**Step 2:** Use your answer to compute

\[
\int_{0}^{\pi/2} x \sin(x^2) \, dx = F\left(\frac{\pi}{2}\right) - F(0).
\]

\[
\int_{0}^{\pi/2} x \sin(x^2) \, dx = -\frac{1}{2} \cos\left(\frac{\pi}{2}\right) - \left(-\frac{1}{2} \cos(0^2)\right) = 1/2
\]
Worksheet: Area between curves

Example 1:
Find the area of the region between \( y = e^x \) and \( y = 1/(1 + x) \) on the interval \([0, 1]\).

1. Check for intersection points (verify algebraically that \( x = 0 \) is the only intersection by setting \( e^x = \frac{1}{x+1} \)).
2. Decide which function is on top (\( f(x) \)) and which function is on bottom (\( g(x) \)).
3. Calculate \( \int_0^1 f(x) - g(x) \, dx \).

Check: What if you get a negative answer?
Example 2:
Find the area of the region bounded by \( y = x^2 - 2x \) and \( y = 4 - x^2 \).

1. Check for intersection points (Solve \( x^2 - 2x = 4 - x^2 \)). There will be two, \( a \) and \( b \); this is where the functions cross.
2. Between this two points, which function is on top (\( f(x) \)) and which function is on bottom (\( g(x) \)).
3. Calculate \( \int_a^b f(x) - g(x) \, dx \).

Check: What if you get a negative answer?
Example 3
Find the area between $\sin x$ and $\cos x$ on $[-3\pi/4, 5\pi/4]$.

(Hint: There are several places where $\sin(x) = \cos(x)$. For example, $x = \pi/4$.)
Example 4
Calculate the area under the curve $y = \arccos(x)$ from $x = 0$ to $x = 1$.

Hint: Since we don’t know $\int \arccos(x) \, dx$, use the fact that $y = \arccos(x)$ if and only if $\cos(y) = x$.
(1) Draw graphs of both $y = \arccos(x)$ and $x = \cos(y)$ on separate axes (the first with $x$ on the horizontal axis, and the second with $y$ on the horizontal axis).
(2) What integral, involving $\cos(y)$ (and endpoints for $y$’s instead of $x$’s, and with a $dy$ instead of a $dx$) will compute the same area as $\int_0^1 \arccos(x) \, dx$?
Example 5
Calculate the area under the curve $y = \arcsin(x)$ from $x = 0$ to $x = 1$.

Hint: Similar to Example 4, but be careful! Be sure to draw the pictures before writing down the corresponding integrals!
Answers
Example 1: \( e - 1 - \ln(2) \)
Example 2: 9

Example 3: \( 4\sqrt{2} \)

Example 4: 1

Example 5: \( \frac{\pi}{2} - 1 \)
Extra practice: Areas using definite integrals

1. Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, the $x$-axis, and the lines $x = 3$ and $x = 4$.

2. Find the area lying below the $x$-axis and above the parabola $y = 4x + x^2$.

3. Graph the curve $y = 2\sqrt{9 - x^2}$ and determine the area enclosed between the curve and the $x$-axis.

4. Find the area bounded by the curve $y = x(x - 3)(x - 5)$, the $x$-axis and the lines $x = 0$ and $x = 5$.

5. Find the area enclosed between the curve $y = \sin 2x$, $0 \leq x \leq \pi/4$ and the axes.

6. Find the area enclosed between the curve $y = \cos 2x$, $0 \leq x \leq \pi/4$ and the axes.

7. Find the area enclosed between the curve $y = 3\cos x$, $0 \leq x \leq \pi/2$ and the axes.

8. Show that the ratio of the areas under the curves $y = \sin x$ and $y = \sin 2x$ between the lines $x = 0$ and $x = \pi/3$ is $2/3$.

9. Find the area enclosed between the curve $y = \cos 3x$, $0 \leq x \leq \pi/6$ and the axes.

10. Find the area enclosed between the curve $y = \tan^2 x$, $0 \leq x \leq \pi/4$ and the axes.

11. Find the area enclosed between the curve $y = \csc^2 x$, $0 \leq x \leq \pi/4$ and the axes.

12. Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.

13. Graph the curve $y = x/\pi + 2\sin^2 x$ and find the area between the $x$-axis, the curve and the lines $x = 0$ and $x = \pi$.

14. Find the area bounded by $y = \sin x$ and the $x$-axis between $x = 0$ and $x = 2\pi$.

15. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

16. Find the area bounded by the curve $y = x(2 - x)$ and the line $x = 2y$.

17. Find the area bounded by the curve $x^2 = 4y$ and the line $x = 4y - 2$.

18. Calculate the area of the region bounded by the parabolas $y = x^2$ and $x = y^2$.

19. Find the area of the region included between the parabola $y^2 = x$ and the line $x + y = 2$.

20. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = x$. 

21. Find the area of the part of the first quadrant which is between the parabola \( y^2 = 3x \) and the circle \( x^2 + y^2 - 6x = 0 \).

22. Find the area of the region between the curves \( y^2 = 4x \) and \( x = 3 \).

23. Use integration to find the area of the triangular region bounded by the lines \( y = 2x + 1 \), \( y = 3x + 1 \) and \( x = 4 \).

24. Find the area bounded by the parabola \( x^2 - 2 = y \) and the line \( x + y = 0 \).

25. Find the area bounded by the curves \( y = 3x - x^2 \) and \( y = x^2 - x \).

26. Graph the curve \( y = (1/2)x^2 + 1 \) and the straight line \( y = x + 1 \) and find the area between the curve and the line.

27. Find the area of the region between the parabolas \( 4y^2 = 9x \) and \( 3x^2 = 16y \).

28. Find the area of the region between the curves \( x^2 + y^2 = 2 \) and \( x = y^2 \).

29. Find the area of the region between the curves \( y = x^2 \) and \( x^2 + 4(y - 1) = 0 \).

30. Find the area of the region between the circles \( x^2 + y^2 = 4 \) and \( (x - 2)^2 + y^2 = 4 \).

31. Find the area of the region enclosed by the parabola \( y^2 = 4ax \) and the line \( y = mx \).

32. Find the area between the parabolas \( y = 4ax \) and \( y^2 = 4ay \).

33. Find the area of the region between the two circles \( x^2 + y^2 = 1 \) and \( (x - 1)^2 + y^2 = 1 \).

34. Find the area bounded by the curves \( y = x \) and \( y = x^3 \).

35. Graph \( y = \sin x \) and \( y = \cos x \) for \( 0 \leq x \leq \pi/2 \) and find the area enclosed by them and the \( x \)-axis.