Modeling with Differential Equations: Introduction to the Issues
Warm-up

Do you know a function . . .

. . . whose first derivative is the same as the function itself, i.e.

\[
\frac{d}{dx} f(x) = f(x) ?
\]

. . . whose first derivative is negative of the function, i.e.

\[
\frac{d}{dx} f(x) = -f(x) ?
\]

. . . whose second derivative is negative of itself, i.e.

\[
\frac{d^2}{dx^2} f(x) = -f(x) ?
\]
Warm-up

Do you know a function…
…whose first derivative is the same as the function itself, i.e.

\[
\frac{d}{dx} f(x) = f(x)?
\]

\[e^x, \quad 2e^x, \quad Ae^x, \quad 0\]

…whose first derivative is negative of the function, i.e.

\[
\frac{d}{dx} f(x) = -f(x)?
\]

\[e^{-x}, \quad 2e^{-x}, \quad Ae^{-x}, \quad 0\]

…whose second derivative is negative of itself, i.e.

\[
\frac{d^2}{dx^2} f(x) = -f(x)\
\]

\[\cos(x), \quad \sin(x), \quad \sin(x) + \cos(x), \quad A\cos(x) + B\sin(x), \quad 0\]
**Goal:**
Given an equation relating a variable (e.g. $x$), a function (e.g. $y$), and its derivatives ($y', y'', \ldots$), **what is $y$?**
i.e. How do I solve for $y$?
Goal:
Given an equation relating a variable (e.g. $x$), a function (e.g. $y$),
and its derivatives ($y'$, $y''$, ...), what is $y$?
i.e. How do I solve for $y$?

Why?
Many physical and biological systems can be modeled with
differential equations. Also, it can be a lot harder to model a
function long term than it is to measure how something changes as
the system goes from one state to another.
Some examples

**Obervation:** The rate of increase of a bacterial culture is proportional to the number of bacteria present at that time.

**Equation:** \[ \frac{dP}{dt} = kP \]

**Solution:** \[ P = Ae^{kt}, \] where \( A \) is a constant.

**Obervation:** The motion of a mass on a spring is given by two opposing forces: (1) the force exerted by the mass in motion \( F = ma = m\frac{d^2}{dt^2}D \) and (2) the force exerted by the spring, proportional to the displacement from equilibrium \( F = kD \).
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**Equation:** \[ m \frac{d^2}{dt^2} D = -kD \]
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**Equation:** \( m \frac{d^2}{dt^2}D = -kD \)

**Solution:** \( D(t) = A \cos(t \sqrt{k/m}) + B \sin(t \sqrt{k/m}) \), where \( A, B, k, \) and \( m \) are all constants.
Slope Fields

If you can write your differential equation like

\[
\frac{dy}{dx} = F(x, y)
\]

then you really have a way of saying

“If I’m standing at the point \((a, b)\),
then I should move from here with slope \(F(a, b)\).”
Slope Fields

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Some examples:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dP}{dt} = kP$$

$$\frac{dx}{dt} = t^2 \sin(xt) + x^2$$
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Some examples:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dP}{dt} = kP$$

$$\frac{dx}{dt} = t^2 \sin(xt) + x^2$$

Some non-examples:

$$\frac{dy}{dx} = -\frac{x}{y} + \frac{d^2y}{dx^2}$$

$$\frac{dP}{dt} * \frac{d^2P}{dt^2} = kP$$

$$m * \frac{d^2D}{dt^2} = -kD$$
\[
\frac{dy}{dx} = -\frac{x}{y}
\]

<table>
<thead>
<tr>
<th>$x$</th>
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Slope field:

Arrows point in the direction of semicircles!

Slope $y = \pm \sqrt{r^2 - x^2}$?

Check:

\[
\frac{dy}{dx} = -\frac{2x}{\pm \sqrt{r^2 - x^2}} = -\frac{x}{y},
\]

Slope field:
\[
\frac{dy}{dx} = -\frac{x}{y}
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Slope field:

Arrows point in the direction of semicircles!

\[ y = \pm \sqrt{r^2 - x^2} \]

Check:

\[ \frac{dy}{dx} \pm \sqrt{r^2 - x^2} = -\frac{2x}{\pm 2\sqrt{r^2 - x^2}} = -\frac{x}{y} \]
\[
\frac{dy}{dx} = \frac{-x}{y}
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Slope field:

Arrows point in the direction of semicircles!

\[y = \pm \sqrt{r^2 - x^2}\]

Check:

\[
\frac{dy}{dx} = \pm \sqrt{r^2 - x^2} = -\frac{x}{y},
\]
\[
\frac{dy}{dx} = -\frac{x}{y}
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Slope field:

Arrow points in the direction of semicircles.

\[y = \pm \sqrt{r^2 - x^2}\]

Check:

\[
\frac{dy}{dx} \pm \sqrt{r^2 - x^2} = -\frac{2x}{2\sqrt{r^2 - x^2}} = -\frac{x}{y},
\]

\[
m = -1
\]
\[ \frac{dy}{dx} = -\frac{x}{y} \]

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Slope field:

Arrows point in the direction of semicircles!

\[ y = \pm \sqrt{r^2 - x^2} \]

Check:

\[ \frac{dy}{dx} \pm \sqrt{r^2 - x^2} = -\frac{x}{y}, \quad m=1 \]
\[
\frac{dy}{dx} = -\frac{x}{y}
\]

<table>
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<tr>
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<th>y</th>
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Arrows point in the direction of semicircles!

\[y = \pm \sqrt{r^2 - x^2}\]

Check:

\[
\frac{d}{dx} \pm \sqrt{r^2 - x^2} = -\frac{2x}{\pm \sqrt{r^2 - x^2}} = -\frac{x}{y},
\]
\[
\begin{array}{c|cc}
 x & y & \frac{dy}{dx} = -\frac{x}{y} \\
 \hline
 0 & 1 & 0 \\
 0 & -1 & 0 \\
 1 & 1 & -1 \\
 1 & -1 & 1 \\
 -1 & 1 & 1 \\
 -1 & -1 & -1 \\
 2 & 1 & -1 \\
 1 & 2 & \text{undefined} \\
 -2 & 0 & \\
\end{array}
\]
\[
\begin{array}{|c|c|c|}
\hline
x & y & \frac{dy}{dx} = -x/y \\
\hline
0 & 1 & 0 \\
0 & -1 & 0 \\
1 & 1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1 \\
-1 & -1 & -1 \\
2 & 1 & -2 \\
1 & 2 & \\
-2 & 0 & \\
\hline
\end{array}
\]

Slope field:

Arrows point in the direction of semicircles!

\[y = \pm \sqrt{r^2 - x^2}\]

Check:

\[\frac{dy}{dx} = \pm \sqrt{r^2 - x^2} = -\frac{x}{y},\]
The slope field for the differential equation \( \frac{dy}{dx} = -\frac{x}{y} \) is shown below. The arrows point in the direction of the semicircles.

### Table

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
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</table>

The slope field indicates that the slope is \(-1/2\) at the indicated point.
\[
\frac{dy}{dx} = -\frac{x}{y}
\]

\[
\begin{array}{ccc}
  x & y & \frac{dy}{dx} = -\frac{x}{y} \\
  0 & 1 & 0 \\
  0 & -1 & 0 \\
  1 & 1 & -1 \\
  1 & -1 & 1 \\
 -1 & 1 & 1 \\
 -1 & -1 & -1 \\
 -1 & -1 & -1 \\
  2 & 1 & -2 \\
  1 & 2 & -1/2 \\
 -2 & 0 & \text{undefined} \\
\end{array}
\]

Slope field:

- Arrows point in the direction of semicircles!

- Check:
  \[
  \frac{dy}{dx} = -\frac{x}{y}
  \]
\[
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</table>

Slope field:

Arrows point in the direction of semicircles!

Check:

\[
dx \pm \sqrt{r^2 - x^2} = -x/y,
\]

\[
Slope = -\frac{x}{y}
\]
\[
\frac{dy}{dx} = -\frac{x}{y}
\]

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
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Slope field:

Arrows point in the direction of semicircles!

Check:

\[
\frac{dy}{dx} = \pm \sqrt{r^2 - x^2}
\]

\[
\frac{dy}{dx} = -\frac{x}{y},
\]
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<tr>
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Slope field:

Arrows point in the direction of semicircles! $y = \pm\sqrt{r^2 - x^2}$

Check:

\[
\frac{dy}{dx} \pm \sqrt{r^2 - x^2} = -\frac{x}{y},
\]
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<thead>
<tr>
<th>x</th>
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**Slope field:**

Arrows point in the direction of semicircles! \( y = \pm \sqrt{r^2 - x^2} \)?
\[ \frac{dy}{dx} = -\frac{x}{y} \]

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Arrows point in the direction of semicircles! \( y = \pm \sqrt{r^2 - x^2} \)?

Check:

\[
\frac{d}{dx} \pm \sqrt{r^2 - x^2} = \frac{-2x}{\pm 2\sqrt{r^2 - x^2}}
\]
\[ \frac{dy}{dx} = -\frac{x}{y} \]

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(\frac{dy}{dx} = -\frac{x}{y})</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<tr>
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<tr>
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<td>-1</td>
</tr>
<tr>
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<td>-2</td>
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<tr>
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<td>2</td>
<td>-1/2</td>
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<tr>
<td>-2</td>
<td>0</td>
<td>undef</td>
</tr>
</tbody>
</table>

Arrows point in the direction of semicircles! \[ y = \pm \sqrt{r^2 - x^2}? \]

Check: \[ \frac{d}{dx} \pm \sqrt{r^2 - x^2} = \frac{-2x}{\pm 2\sqrt{r^2 - x^2}} = -\frac{x}{y} \]
Solving explicitly (get a formula!)

We’ve done…

1. Get lucky
   “what’s a function you know whose derivative blah blah …”

2. Differential equations of the form

\[
\frac{dy}{dx} = f(x)
\]

*Find the antiderivative!*

Today, we’ll add

3. Differential equations of the form

\[
\frac{dy}{dx} = f(x) \times g(y)
\]

*Use "Separation of Variables"*
A separable differential equation is one of the form

\[ \frac{dy}{dx} = f(x) \cdot g(y). \]
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\]

Some examples:

\[
\frac{dy}{dx} = -\frac{x}{y} = (-x) \cdot \left(\frac{1}{y}\right)
\]

\[
\frac{dx}{dt} = t^2 \sec(x)
\]
Separable Equations

A separable differential equation is one of the form

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Some examples:

Some non-examples:

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\[ \frac{dy}{dx} = x + y \]

\[ \frac{dx}{dt} = \frac{t + x}{xt^2} \]
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Some non-examples:

\[ \frac{dy}{dx} = x + y \]
\[ \frac{dx}{dt} = \frac{t + x}{xt^2} \]

A separable equation is one in which we can put all of the \( y \)'s and \( dy \)'s (as products) on one side of the equation and all of the \( x \)'s and \( dx \)'s (as products) on the other...
Examples

(1) If \( \frac{dy}{dx} = -\frac{x}{y} \), then \( y \ dy = -x \ dx \).
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To solve (1), integrate both sides:

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So

\[
y = \pm \sqrt{2(-x^2/2 + c_1 - c_2)} = \pm \sqrt{a - x^2}
\]

where \( a = 2(c_1 - c_2) \).
Examples

Slope field for $\frac{dy}{dx} = -\frac{x}{y}$:

Suggested and checked $y = \pm \sqrt{r^2 - x^2}$
Examples

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*Find an implicit formula for (2) (with no derivatives left in it)*
How many solutions are there?

**Existence?**
How do I know I even get a solution?
An important result in the theory of differential equations is **Peano’s Existence Theorem**, which states...

If \( \frac{dy}{dx} = F(x, y) \) and \( y(a) = b \),
where \( F(x, y) \) is continuous in a domain \( D \),
then there is always at least one solution in the domain, and any such solution is differentiable.

**Uniqueness?**
How do we know that there is not another solution?

If, additionally, \( F(x, y) = f(x)g(y) \), and if \( g' \) and \( f' \) are continuous, then solution is unique.
Example of non-uniqueness

Suppose $\frac{dy}{dx} = \frac{1}{x}$ and $y(2) = 1$. Since $1/x$ is not continuous at $x = 0$, we might have lots of solutions, all that split at 0!
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(1) Match the differential equations to the slope fields:

- (A) \( \frac{dy}{dx} = \frac{1}{5} xy \)
- (B) \( \frac{dy}{dx} = x + y \)
- (C) \( \frac{dy}{dx} = \cos(x) \)
- (D) \( \frac{dy}{dx} = \cos(y) \)

(a) (b) (c) (d)

(2) Solve the initial value problems

- (a) \( \frac{dy}{dx} = \frac{1}{5} xy, \quad y(0) = 2; \)
- (b) \( \frac{dy}{dx} = \sin(x)/y^2, \quad y(0) = 3. \)