Worksheet – The tangent line problem

We’ve been building towards studying rates of change, e.g.

- rate at which position changes versus time (velocity);
- rate at which birthrate changes versus average household income;
- rate at which profit margin changes versus production volume.

In general, the instantaneous rate of change of a function \( f(x) \) versus \( x \) at a point \( a \) is given by the limit of the difference quotient:

\[
\text{inst. rate of change} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

Another word for the instantaneous rate of change of a function \( f(x) \) at a point \( a \) is the derivative of \( f(x) \) at \( x = a \), written \( f'(a) \). So

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

The derivative also has a geometric interpretation:

\[
f'(a) = \text{slope of the line tangent to } y = f(x) \text{ at } x = a.
\]

Example 1: Below is a graph of the function \( f(x) = \sqrt{1 - x^2} \) (the half circle with radius 1). Without calculating any limits, what is

(a) \( f'(0) \)?

(b) \( f'(\frac{\sqrt{2}}{2}) \)?

(c) \( f'(-\frac{\sqrt{2}}{2}) \)?

[hint: for (b) and (c), draw a line from the origin to the point in question. What angle does that make with the x-axis? What is the slope of that line? For a circle, the line tangent at a point is perpendicular to the ray from the center to the point.]
Once you have the slope, it’s pretty easy to write down the equations for the tangent line using point-slope form:

\[ y = m(x - x_0) + y_0 \quad \text{becomes} \quad y = f'(a)(x - a) + f(a). \]

**Example 2:** What is the equation for the line tangent to \( f(x) = \sqrt{1-x^2} \) at 
(a) \( x = 0 \)?

(b) \( x = \frac{\sqrt{2}}{2} \)?

(c) \( x = -\frac{\sqrt{2}}{2} \)?

Check your answers by first sketching the lines you wrote down in (a)-(c), and then sketching the function \( f(x) = \sqrt{1-x^2} \) on the axes to the right.

**Example 3:** For reference, the graph of \( f(x) = \sin(x) \) is:

(a) The function \( \sin(x) \) has infinitely many points \( x = a \) where \( f'(a) = 0 \). What are they?

(b) There are exactly two horizontal lines which are tangent to \( \sin(x) \). What are they?

(c) [Bonus] Can you think of a function which has infinitely many points where \( f'(a) = 0 \), not just anywhere, but between \( x = 0 \) and \( x = \pi \)? [hint: think back to the day we did limits. There is some function \( g(x) \) which we could plug into \( \sin(x) \) which will make \( \sin(g(x)) \) a good answer to this question.]

Answers: 1(a) : 0, (b) : -1, (c) : 1, 2(a) : \( y = 1 \), (b) : \( y = -x + \sqrt{2} \), (c) : \( x + \sqrt{2} \), 3(a) : \( \frac{\pi}{2} + \pi k \), (b) : \( y = \pm 1 \).
Calculating derivative using limits

Recall from last Friday that we have a few tricks for calculating limits \( \lim_{x \to a} g(x) \):

1. **Plugging in:** If \( g(x) \) is continuous, and \( g(a) \) is defined, then \( \lim_{x \to a} g(x) = g(a) \).
   
   For example, \( \lim_{x \to 2} \frac{x + 1}{x^2 - 3} = \)

2. **Factor and cancel:** If \( g(x) \) is rational, and \( g(a) \) is not defined, but \( a \) is a root of the numerator and denominator, then factor and cancel:
   
   For example, \( \lim_{x \to 2} \frac{x + 1}{x - 2} \) is undefined,
   
   but \( \lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \)

3. **Expand and cancel:** It’s like spring cleaning – make a mess, and then clean up!.
   
   For example, since \((x + 2)^3 = x^3 + 6x^2 + 12x + 8\),
   
   \( \lim_{x \to 0} \frac{x}{(x + 2)^3 - 8} = \)
4. **Common denominators:** If you have a sum or difference of fractions, find a common denominator and see what happens.

For example,

\[
\lim_{x \to 0} \left( \frac{1}{x} - \frac{1}{x(x+1)} \right) = \text{[ ]}
\]

5. **Multiply top and bottom by the conjugate:** If you have a difference of square roots (like \(\sqrt{a} - \sqrt{b}\)), you can multiply and divide by the conjugate, \(\sqrt{a} + \sqrt{b}\). This is useful because

\[
(\sqrt{a} - \sqrt{b}) (\sqrt{a} + \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b
\]

For example, try multiplying by \(\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}\): (notice \(2 = \sqrt{4}\))

\[
\lim_{x \to 0} \frac{x}{\sqrt{x+4} - 2} = \text{[ ]}
\]

Answers: 1 : -3, 2 : \(\frac{1}{4}\), 3 : \(\frac{1}{17}\), 4 : 1, 5 : 4.
Now that we have these tools, let’s calculate some derivatives!

(A) Use the limit definitions the derivative of \( f(x) = x^2 \) at \( x = 1 \):

\[
\begin{align*}
    f'(1) &= \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} \\
    &= \lim_{h \to 0} \frac{(1 + h)^2 - (1)^2}{h} \\
    &= \text{[compute]} \\
\end{align*}
\]

(B) Use the limit definitions the derivative of \( f(x) = x^3 \) at \( x = -2 \):

\[
\begin{align*}
    f'(-2) &= \lim_{h \to 0} \frac{f(-2 + h) - f(-2)}{h} \\
    &= \lim_{h \to 0} \frac{(-2 + h)^3 - (-2)^3}{h} \\
    &= \text{[compute]} \quad \text{careful! } (-2)^3 = -8, \text{ so } -(\text{-2})^3 = 8 \\
\end{align*}
\]
(C) Use the limit definitions the derivative of \( f(x) = \frac{1}{x} \) at \( x = 3 \):

\[
f'(3) = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}
\]

= 

(D) Use the limit definitions the derivative of \( f(x) = \sqrt{x} \) at \( x = 5 \):

\[
f'(3) = \lim_{h \to 0} \frac{f(5 + h) - f(5)}{h}
\]

= 

Answers: A : 2, B : 12, C : \(-\frac{1}{9}\), D : \(\frac{1}{2\sqrt{5}}\)
Back to tangent line equations:

Use your answers to A-D on the previous two pages to calculate the lines tangent to

(a) \( f(x) = x^2 \) at \( x = 1 \)

(b) \( f(x) = x^3 \) at \( x = -2 \)

(c) \( f(x) = \frac{1}{x} \) at \( x = 3 \)

(d) \( f(x) = \sqrt{x} \) at \( x = 5 \)

Check your answers by sketching the lines from (a)-(d) on onto the appropriate graphs below:

Answers: a : \( y = 2x - 1 \),  b : \( y = 12x + 16 \),  c : \(- \frac{1}{5} x + \frac{2}{5} \),  d : \( \frac{1}{2\sqrt{5}} x + \frac{\sqrt{5}}{2} \)
When can we take derivatives?
Not all functions have derivatives at all places. Before calculating $f'(a)$, first ask . . .

1. Is $f(x)$ defined at $x = a$?

For example, even if it looks like you could draw a tangent line, if there’s a hole, $f'(a)$ does not exist!

(It’s tempting to say $f'(a)$ exists here in part because $f(x)$ has a continuous extension at $a$.)

2. Is $f(x)$ continuous at $x = a$?

For example, even if it looks like you could draw a tangent line, if there’s a jump, $f'(a)$ does not exist!

(Try drawing just one line that is tangent to that isolated point. It’s tempting to say $f'(a)$ exists here in part because $f(x)$ has a removable discontinuity at $a$.)

Again, even if the slope looks the same from the left and from the right, if there’s a discontinuity, $f'(a)$ does not exist!

3. Is there a “corner” at $x = a$?

Next we’ll explore how to find these algebraically, but if there’s a sharp corner at $x = a$, then $f'(a)$ does not exist!

(Try drawing just one line that is tangent to that corner)
What’s wrong with corners?

Let \( f(x) = \begin{cases} 
  x^2 & x < 2, \\
  x + 2 & x \geq 2. 
\end{cases} \)

(a) Verify that \( f(x) \) is continuous at \( x = 2 \).

(b) Sketch a graph of \( f(x) \).
(c) Estimate, and then calculate the right sided derivative.

(i) Estimate:

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<tr>
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<th>$f(2 + h) - f(2)$</th>
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(ii) Explain why $\lim_{h \to 0^+} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2 + 2 + h) - (2 + 2)}{h}$.

(iii) Calculate $\lim_{h \to 0^+} \frac{f(2 + h) - f(2)}{h}$. 
(d) Estimate, and then calculate the left sided derivative. (OK to use a calculator for (i))

(i) Estimate:

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(ii) Explain why \( \lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h} = \lim_{h \to 0^-} \frac{(2 + h)^2 - (2)^2}{h} \).

(iii) Calculate \( \lim_{h \to 0^-} \frac{f(2 + h) - f(2)}{h} \).

(e) Compare your answers to (b) and (c), and explain why \( \lim_{h \to 0} \frac{f(2 + h) - f(2)}{h} \) does not exist. Explain why \( f'(2) \) does not exist.
(f) Sketch graphs of the following functions and identify points where each function is not differentiable:

\[ f(x) = |x| \quad g(x) = |x-2| \quad h(x) = |4-|x-2|| \quad \psi(x) = \frac{|x|}{x} \quad \phi(x) = \begin{cases} x^2 & x < 0, \\ x^4 & x \geq 0. \end{cases} \]

[Hint: for \( h(x) \), start by plotting some points, and then find points where \( x - 2 \) goes from positive to negative, and where \( 4 - |x - 2| \) goes from positive to negative.]
Answers: a: check each requirement,  
       c (iii): 1,  
       d (iii): 4,  
       e: do the two sides meet?

   \[ f(x) : x = 0, \quad g(x) : x = 2, \quad h(x) : x = -2, 2, 6, \quad \psi(x) : x = 0, \quad \phi(x) : \text{no } x! \]