Math 43 Homework 3

2.5 #6 \((UV)_{xx} = U_{xx}V + 2U_{x}V_{x} + UV_{xx}\)
\((UV)_{yy} = U_{yy}V + 2U_{y}V_{y} + UV_{yy}\)
Use C-R to show \((UV)_{xx} + (UV)_{yy} = 0\)

3.1 #4 \(p(z) = (z - \lambda_1) \cdots (z - \lambda_n)\) where \(\lambda_1, \ldots, \lambda_n\) are roots (possibly repetitions)
\(\alpha_0 = (-1)^n \lambda_1 \cdots \lambda_n\)
\(|\lambda_1 \cdots \lambda_n| = |\alpha_0| > 1\)
\(|\lambda_1| \cdots |\lambda_n| > 1\) if all \(|\lambda_i| \leq 1\), then \(|\lambda_1| \cdots |\lambda_n| < 1\).
Impossible.

#7 \(p(z) = (z - z_1)^m q(z)\) for some polynomial \(q(z)\)
Take derivative: \(p'(z) = m(z - z_1)^{m-1} q(z) + (z - z_1)^m q'(z)\)
\(= (z - z_1)^{m-1} (mq(z) + (z - z_1)q'(z))\)
\(\therefore p'(z)\) has a root of multiplicity \(m-1\)

3.2 #4 \(\exp(i \pi r + i \theta) = \exp(i \pi r) \exp(i \theta) = r e^{i \theta} = W\)

#7 \(\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}\)

Add them.

#9 \(\frac{\sinh z}{z} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!}\)

#11 \(e^{z}/z\) is entire (since \(e^{z}\) is never 0)
\(\therefore \Re\left(\frac{e^{z}}{z}\right)\) is harmonic in the plane

#20 \(W = e^z\) fix \(-1 \leq x_0 \leq 1\) and let \(0 \leq y \leq \pi\) and \(z = x_0 + iy\) Then
\[e^z = (e^{x_0} \cos y, e^{x_0} \sin y)\]. This point lies on the upper semicircle center 0 and radius \(e^{x_0}\). The union of all of these semicircles from those of radius \(e^{-1}\) \((x_0 = -1)\) to those of radius \(e\) \((x_0 = 1)\) is the shaded area.
3.3 #11 Since $z^2 + 2z + 3 = 2$ when $z = -1$, we want a log function $L_x$ differentiable at $z$. Any $x = (k + 1)i\pi$ will do, say $x = -\pi$, so $L_x = \log$. The function is 
$\log (z^2 + 2z + 3)$.

#14 Assume $P(z)$ exists $\Rightarrow F'(z) = (\log z)'$ for all $z \in D$.

$z$ not a negative real number (Since $\log z$ not differentiable there). $F(z) = \log z + C$ ($C$ complex constant) for every $z \in D$.

not a negative real number. Let $z_0$ be on the negative real axis. Then $\lim \log z_0$ does not exist. But $F(z)$ is analytic on $D$, hence continuous on $D$. Therefore $P(z) - C$ is continuous on $D$. $\Rightarrow \lim (F(z) - C)$ exists.

Contradiction.