Homework 7

8.1 \( e^{2z} \)

5 \( s_n - s_{n-1} = c_n \), \( \lim_{n \to \infty} (s_m - s_{m-1}) = \lim s_m - \lim s_{m-1} = 1 - 1 = 0 \)

6 If \( \sum c_i \) converges, then \( \sum c_i \to 0 \). But if \( |c| > 1 \), then \( |c| \to \infty \) (as \( |c| \to 1 \)). So \( \sum c_i \to 0 \). \( \sum c_i \) diverges.

7a Ratio Test

\[ \frac{|d_n - 0|}{d_{n+1}} \leq \frac{1}{k^n} \Rightarrow \sum \frac{1}{k^n} \text{ converges } \Rightarrow \text{ given series converges} \]

14b \( \left| \frac{k^n}{k^n} \right| \leq \frac{k^n}{k^n} \leq \frac{1}{k^n} \Rightarrow \text{ given series converges} \)

14c \( \left| \frac{k^n}{k^{n+1}} \right| \leq \frac{k^n}{k^n} = \frac{1}{k} \Rightarrow \text{ given series converges} \)

16 with partial sum

\( s_n = (z_n - 2z) + (z_n - 2z) + \cdots + (z_n - 2z) = z_{n+1} - 2z \)

Series converges \( \iff \) \( s_n \) converges \( \iff \) \( z_n \) converges

5.2

16 Done in class

16 \( f(z) = (1 - z)^{-1} \), \( f'(z) = (-1)z^{-2} \), \( f''(z) = 2!(-1)z^{-3} \) \( \cdots \)

\( f^{(m)}(z) = m!(-1)(z^{-m+1}) \)

\( a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{(1-n)^{n+1}} \) Series \( \sum \frac{1}{(1-i)^{n+1}} \)

2b All \( z \)

2d Singularity at \( z = 1 \) The circle around \( i \) with radius \( \sqrt{2} \) is the largest circle around \( i \) with no singularity in its interior. \( |z-i| < \sqrt{2} \)

4 By taking \( (1+z)^x = e^{x \log(1+z)} \) we have selected a branch

\( \frac{d}{dz} \frac{1}{z} \frac{d}{dz} (1+z)^x = (1+2z) \frac{x}{1+z} = (1+z)^{x-1} \cdot \alpha \)

\( \frac{d}{dz} (1+z)^x = \alpha (x-1)(1+z)^{x-2} \) et cetera

11b \( (1+z^2 + \frac{z^3}{3} + \cdots) = (z-1)(a_0 + a_1 z + a_2 z^2 + \cdots) \)
Solve for $a_n$,

\[ f(z) = \sum a_n (z-z_0)^n \quad \text{with} \quad a_n = \frac{f^{(n)}(z_0)}{n!} = 0 \quad \text{for every} \quad z, \quad |z-z_0| < R. \]

6.3 Suppose series converges at $z = e^{it}$ some fixed $t$. Then the series is $\sum c_n e^{it}$. If this converges then the $n$th term $\to 0$. But $\lim_{n \to \infty} |\text{le}v\text{e}m| = 1$.

\[ 3^6 \frac{|3-i|^{1/k_n} k^2}{(k+1)^2/3-1} \left(k^2 \right)^{1/k_n} = \frac{|3-i|^{1/k_n}}{(k+1)^2} \to 10, \quad R = \frac{1}{\sqrt{10}} \]

Circle center $-2$, radius $\frac{1}{\sqrt{10}}$.

4. No. $|z+z_0| = \sqrt{13}$ so radius of convergence $< \sqrt{13}$.

5. $a_n = \frac{63}{36}, a_6 = \frac{63(6)}{6!}, a_6 = \frac{63(6)}{6!} = \frac{63(6)}{6!}$

5b $f(z) = \frac{4}{3} - \frac{3}{3} z^2 + \frac{3}{3} z^3 + \frac{4}{3} z^4 + \cdots$

Divide by $z^4$ and integrate term-by-term. The only non-zero integral is $\int \frac{3}{3} \frac{1}{z^4} \, dz = 2 + i$.

6a By dividing the Maclaurin series for $f(z)$ by $z$ we get $f(z) = 1 - \frac{z^4}{3} + \frac{z^4}{5} - \cdots$ for $|z| < R$.

But $f(z)$ equals above series when $z = 0$.

(b) By Theorem 10, $f(z)$ is analytic everywhere.

(c) $f^{(n)}(0) = n! a_n$ where $a_n$ is the $n$th coefficient of the series.

10. Because the power series in the Maclaurin series of some function and a Maclaurin series can be differentiated term-by-term with the same disk of convergence. (Thm 4)

* Thm: 41