(1) Compute the following integrals.

(a) \[ \int_C \frac{1}{z(z-2)^2} \, dz \], where \( C \) is the circle \( |z| = 4 \).

(b) \[ \int_C \frac{1}{z(z-2)^2} \, dz \], where \( C \) is the circle \( |z| = 1 \).

(c) \[ \int_C \frac{e^z}{z^2 + 1} \, dz \], where \( C \) is the square with vertices \( \pm 2 \pm 2i \).

Solution.

(a) The easiest way to solve this problem is to use the residue formula. We have \( f(z) = 1/z(z-2)^2 \), which is holomorphic everywhere except singularities at \( z = 0, 2 \). Also, these singularities are poles, since \( 1/f(z) = z(z-2)^2 \) satisfies \( \lim_{z \to 0} 1/f(z) = 0, \lim_{z \to 2} 1/f(z) = 0 \). We easily see that \( z = 0 \) is a pole of order 1 and \( z = 2 \) is a pole of order 2.

To use the residue formula we need to compute the residues of these two poles. For \( z = 0 \), since this is a pole of order 1, the residue of \( f \) at this pole is

\[ \lim_{z \to 0} (z - 0)f(z) = \lim_{z \to 0} \frac{1}{(z - 2)^2} = \frac{1}{4}. \]

The residue of the pole \( z = 2 \) is given by

\[ \lim_{z \to 2} \frac{d}{dz} \frac{1}{z-2} f(z) = \lim_{z \to 2} \frac{d}{dz} \frac{1}{z} = \lim_{z \to 2} \frac{-1}{z^2} = -\frac{1}{4}. \]

Since the curve \( |z| = 4 \) is a circle which contains both poles \( z = 0, 2 \), the residue formula implies

\[ \int_C \frac{1}{z(z-2)^2} \, dz = 2\pi i (\operatorname{res}_{z=0} f(z) + \operatorname{res}_{z=2} f(z)) = 2\pi i (1/4 - 1/4) = 0. \]

(b) This problem is identical to the previous problem, except now the circle \( |z| = 1 \) only encloses the pole \( z = 0 \). Therefore the residue formula implies

\[ \int_{|z|=1} \frac{1}{z(z-2)^2} \, dz = 2\pi i (\operatorname{res}_{z=0} f(z)) = \frac{\pi i}{2}. \]

(c) We can still apply the residue formula to \( C \) and \( f(z) \), even though \( C \) is a square, since this is one of the shapes covered by the version of the residue formula we discussed. (The book calls this an example of a toy contour.) The poles of \( e^z/(z^2 + 1) \) occur at the roots of \( z^2 + 1 \), which are \( \pm i \). It is easy to see that both \( z = i, -i \) are poles of order 1. The residue at \( z = i \) is
The maximum modulus principle applied to \( f \) achieves its absolute maximum on \(|z| = 1\) on the unit circle \(|z| = 1\), but \(|1/f(z)| = 1\) if \(|z| = 1\), so \(|1/f(z)| \leq 1\) for \(|z| < 1\). This implies that \(|f(z)| \geq 1\) if \(|z| < 1\).

But the maximum modulus principle applied to \( f \) implies that \(|f(z)| \leq 1\) for \(|z| < 1\); therefore \(|f(z)| = 1\) for all \(|z| < 1\). From a previous homework assignment we saw that this implies \( f \) is constant, a contradiction to the assumption that \( f \) is not constant in the unit disc. Therefore \( f(z) = 0 \) does have a solution in \(|z| < 1\).

We now show that \( f(z) = 0 \) has a solution when \(|z| < 1\). By way of contradiction, suppose \( f(z) = 0 \) has no solutions in \(|z| < 1\). Then \( 1/f \) will be a holomorphic function on an open set containing the closed unit disc.

For the second part, where we know that \(|f(z)| \geq 1\) when \(|z| = 1\), and there exists a point \( z_0 \) with \(|f(z_0)| < 1\) and \(|z_0| < 1\), we apply a similar idea. We can still apply Rouche’s Theorem to conclude that it is sufficient to show that \( f(z) \) has a root in \(|z| < 1\), since \(|f(z)| \geq 1 > |w|\) for all \(|z| = 1\) and any \(|w| < 1\). To see that \( f(z) \) has a root in the open unit disc, if not, then \( 1/f \) is holomorphic on an open set containing the closed unit disc, so the maximum modulus principle applied to \( 1/f \) implies that the maximum for \(|1/f| \) on \(|z| \leq 1\) occurs on \(|z| = 1\). But on this boundary, \(|1/f(z)| \leq 1\) when \(|z| = 1\), contradicting the fact that \(|1/f(z_0)| > 1\). Therefore \( f(z) \) has a root in \(|z| < 1\), as desired. \( \square \)

(3) For each of the following functions, compute the number of zeros inside the indicated region (zeros should be counted with multiplicity, of course). Your answer should not rely on any machine calculations.

(a) \( f(z) = z^5 - 2z^2 + z - 3 \) inside \(|z| \leq 2\).
(b) \( f(z) = z^8 - 6z^3 + 3z + 1 \) inside \(|z| > 1\).
(c) \( f(z) = 3e^z - z^2 \) inside \(|z| < 1\).
Solution.

(a) Apply Rouche’s Theorem with \( f(z) = z^5, g(z) = -2z^2 + z - 3 \). On the boundary circle \( |z| = 2 \), we have \( |f(z)| = 32, |g(z)| \leq |2z^2| + |z| + 3 = 13 \), so \( |f(z)| > |g(z)| \) on this boundary. Then \( f = z^5, f + g = z^5 - 2z^2 + z - 3 \) have the same number of zeros inside \( |z| \leq 2 \); namely, 5. (Notice there are no roots of \( z^5 - 2z^2 + z - 3 \) on \( |z| = 2 \), since \( |z^5 - 2z^2 + z - 3| \geq 32 - 13 = 19 \) for such \( z \).)

(b) We use Rouche’s Theorem to find the number of roots of the given polynomial inside \( |z| < 1 \). (Notice this is not quite the set we are interested in.) More specifically, let \( f(z) = -6z^3, g(z) = z^8 + 3z + 1 \). Then on \( |z| = 1 \), we have \( |f(z)| = | -6z^3| = 6, |g(z)| \leq |z^8| + |3z| + 1 = 5 \), so \(-6z^3, z^8 - 6z^3 + 3z + 1\) have the same number of roots inside \( |z| < 1 \); namely, 3. Also, notice that \( z^8 - 6z^3 + 3z + 1 \) has no zeros on \( |z| = 1 \), since \( |z^8 - 6z^3 + 3z + 1| \geq 1 \). Finally, the fundamental theorem of algebra and a little elementary algebra implies that \( z^8 - 6z^3 + 3z + 1 \) has 8 roots on all of \( \mathbb{C} \), so the number of roots of \( |z| > 1 \) is \( 8 - 3 = 5 \).

(c) We claim there are no zeros of \( 3e^z - z^2 \) inside \( |z| < 1 \). As a matter of fact we can show this without using Rouche’s Theorem. For if \( 3e^z - z^2 = 0 \), then \( 3e^z = z^2 \), so \( |3e^z| = |z^2| \). However, if \( |z| < 1 \), then \( |z^2| < 1 \), while \( |3e^z| = 3e^{Re(z)} \). Since \( e^{Re(z)} \) is bounded from below by \( e^{-1} \) for \( |z| < 1 \), we have \( |3e^z| > 3/e \) for \( |z| < 1 \). However, \( 3/e > 1 \), so it is impossible for \( |3e^z| = |z^2| \) when \( |z| < 1 \). □

(4) Stein and Shakarchi, Exercise #18, Chapter 3, page 106.

Solution. We will be fairly vague with the solution, although it is easy to make the following solution more precise. Draw a line segment from the small inner circle to the outer circle, and call the point at which this segment meets the outer circle \( z_0 \), say. Then the integral over the inner circle is equal to the integral over the path from \( z_0 \) to the inner circle, across the inner circle, then back across the segment to \( z_0 \). We want to deform the outer circle to the modified inner circle path, which we will call \( C' \), without crossing the point \( z \). One can see that this is possible intuitively; pinch a small portion of the outer circle around \( z_0 \) to the line segment connecting the outer to the inner circle, and then shrink the rest of the outer circle to the inner circle. It is not difficult (a good exercise) to make this rigorous.

The rest of the proof proceeds as in the textbook; in particular, the hint in the book that tells us to notice that \( f(\zeta) - f(z)/(\zeta - z) \) is bounded is true because this is bounded away from \( \zeta = z \) (being a continuous function away from \( \zeta = z \)), and as \( \zeta \to z \) this function approaches \( f'(z) \), so this quotient is also bounded near \( \zeta = z \). □

(5) Stein and Shakarchi, Exercise #22, Chapter 3, page 108. (Notice that you may not be able to analytically continue \( f \) beyond the unit circle, which means you may not be able to apply Cauchy’s Theorem.)
Solution. Suppose it is possible to continuously extend \( f(z) \) to \(|z| = 1\) such that \( f(z) = 1/z \) when \(|z| = 1\). We want to find a contradiction. Let \( C_r \) be the circle of radius \( r \) centered at the origin; then \( \int_{C_1} 1/z \, dz = 2\pi i \), while \( \int_{C_r} f(z) \, dz = 0 \) if \( r < 1 \). Indeed, we can apply Cauchy’s Theorem for the integral over \( C_r \).

Intuitively, this should be a contradiction, because \( f(z) \) is continuously extended to \(|z| = 1\), so that we should expect \( \int_{C_r} f(z) \, dz \) to be continuous in \( r \); in particular, this should be continuous from the left at \( r = 1 \). We will prove this rigorously. Since the closed unit circle is compact, and \( f(z) \) is continuous on this compact set, \( f(z) \) is uniformly continuous. (See the solutions to a HW6, problem 2a, for more information on uniform continuity if necessary.) Let \( e^{it} \) be an arbitrary point on \(|z| = 1\). By the definition of uniform continuity, given any \( \varepsilon > 0 \), there exists a \( \delta \) such that if \(|z_1 - z_2| < \delta\), then \(|f(z_1) - f(z_2)| < \varepsilon\). Choose \( \varepsilon \) small; say, \( \varepsilon = 1/100 \). Apply this with \( z_1 = e^{it} \) and \( z_2 = re^{it} \), where \( r \) is close to 1. (Close enough so that \(|e^{it} - re^{it}| = 1 - r < \delta\).)

On the one hand, we have

\[
\int_{C_1} f(z) \, dz = \int_{0}^{2\pi} f(e^{it})i e^{it} \, dt.
\]

On the other hand,

\[
\int_{C_r} f(z) \, dz = \int_{0}^{2\pi} f(re^{it})i e^{it} \, dt = r \int_{0}^{2\pi} f(re^{it})i e^{it} \, dt.
\]

Notice that since \( \int_{C_r} f(z) \, dz = 0 \), we can the above by \( r \) and still have

\[
0 = \int_{0}^{2\pi} f(re^{it})i e^{it} \, dt.
\]

Let \( g(t) = f(e^{it}) - f(re^{it}) \). By uniform continuity, notice that \(|g(t)| < \varepsilon\). Then \( f(re^{it}) + g(t) = f(e^{it}) \), so

\[
2\pi i = \int_{0}^{2\pi} f(e^{it})i e^{it} \, dt = \int_{0}^{2\pi} (f(re^{it}) + g(t))i e^{it} \, dt.
\]

On the other hand, we already know that \( \int_{0}^{2\pi} f(re^{it})i e^{it} \, dt = 0 \), so

\[
2\pi i = \int_{0}^{2\pi} g(t)i e^{it} \, dt.
\]

Recall that uniform continuity implies \(|g(t)i e^{it}| = |g(t)| < \varepsilon\). Therefore the ML-lemma (strictly speaking, a small variant of the ML-lemma) tells us that

\[
2\pi = \left| \int_{0}^{2\pi} g(t)i e^{it} \, dt \right| \leq \int_{0}^{2\pi} |g(t)i e^{it}| \, dt = \int_{0}^{2\pi} |g(t)| \, dt \leq 2\pi \varepsilon.
\]

However, this is a contradiction as we can choose \( \varepsilon \) arbitrarily small. \( \square \)