AN INTRODUCTION TO COMPLEX ANALYSIS

Contents

1. How is real analysis and complex analysis different? 1
2. Applications of complex analysis 3

What is complex analysis? If forced to give a one-sentence description, many mathematicians would probably say something along the lines of 'calculus for functions of a complex variable'. More precisely, let $\mathbb{C}$ be the usual complex numbers (ie, numbers of the form $a + bi$, where $a, b \in \mathbb{R}, i = \sqrt{-1}$). Then complex analysis can be broadly thought of as calculus for functions $f : \mathbb{C} \to \mathbb{C}$, or perhaps $f : D \to \mathbb{C}$ where D is a subset of the complex numbers.

This is in contrast to real analysis, which is the study of functions $f : \mathbb{R} \to \mathbb{R}$, or more generally $f : \mathbb{R}^n \to \mathbb{R}^m$. Of course, we will mimic many of the definitions that appear in real analysis in the complex setting, but one of the main themes of this course is that complex functions which are differentiable (which, in the complex case, is called *holomorphic*) exhibit many properties which are not satisfied by their real counterparts.

1. How is real analysis and complex analysis different?

How exactly are real analysis and complex analysis different? After all, we can identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, by mapping $a + bi$ to the ordered pair (a, b). So how is the study of functions $f : \mathbb{C} \to \mathbb{C}$ any different than the study of functions $f : \mathbb{R}^2 \to \mathbb{R}^2$? We will spend a fair amount of time explaining the answer in a precise, rigorous fashion, but it is already possible to give a plausible reason why functions $f : \mathbb{C} \to \mathbb{C}$ might exhibit special properties which are not necessarily true of functions $f : \mathbb{R}^2 \to \mathbb{R}^2$.

More specifically, let $f : D \to \mathbb{C}$ be a function where D is an (open) subset of $\mathbb{C}$. Let $z \in D$. Then we say that f is *holomorphic* at z if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists, and if this limit exists, we call this limit the *derivative* of f at z, and write this value as $f'(z)$. Notice that this definition is identical to the definition which is used for functions $f : \mathbb{R} \to \mathbb{R}$ in single-variable calculus. However, the difference here is that the limit $h \to 0$ is a limit in the complex plane, not the real line. The additional multiplicative structure on $\mathbb{C}$ which is not present in $\mathbb{R}^2$ turns out to make all the difference in the properties of holomorphic functions.

Just how different are holomorphic functions from their real counterparts? One of the main themes in this class (really, the primary theme) will be exploring the unusually high level of *regularity* and *rigidity* of holomorphic functions. What do these two terms mean?

Intuitively speaking, the regularity of a function refers to how smooth the function is. For example, let $f : \mathbb{R} \to \mathbb{R}$ be a function. Perhaps f is continuous on $\mathbb{R}$ but not differentiable on all of $\mathbb{R}$; for example, $f(x) = |x|$ is such a function. Even though the function has some regularity (the fact that it is differentiable), it is not as 'smooth' as functions can be: the fact that $f(x) = |x|$ is not differentiable at 0 manifests itself in the fact that the graph of $f(x) = |x|$ is 'jaggy', or not smooth, at $x = 0$. In general, it is possible for a function $f : \mathbb{R} \to \mathbb{R}$ to be k times continuously differentiable, which means that f has k derivatives, and that $f^{(k)} : \mathbb{R} \to \mathbb{R}$ is continuous. If f satisfies this property, we say that f is in $C^k(\mathbb{R})$. Intuitively, then, functions which are in C^k for large k might be smoother than functions which are only in C^k for small k. Of course, functions which are infinitely differentiable (that is, are in C^k for every k) are very smooth. Functions which satisfy this property are in $C^\infty(\mathbb{R})$.

Actually, there does exist a class of functions which exhibit even stronger smoothness properties than functions in C^∞. Recall from calculus that if a function is in C^∞, then we can find the *power series* expansion of $f(x)$ about a point $x = a$ using Taylor's formula. A question which is glossed over in many calculus questions is whether the power series expansion converges to the actual function $f(x)$ in its radius of convergence. (On the other hand, questions like 'what is the radius of convergence of this power series' are beaten endlessly to death in calculus classes!) It turns out that it is entirely possible for a function to have a convergent power series expansion at $x = a$ with radius of convergence $R > 0$, yet for that series to not converge to $f(x)$ anywhere except at the point of expansion $x = a$. This is a somewhat undesirable property. We call functions $f(x)$ whose power series expansion at $x = a$ converges to $f(x)$ in some open interval about a *analytic* at $x = a$.

So here is the situation with real-valued functions: it is entirely possible for a function $f : \mathbb{R} \to \mathbb{R}$ to be, say, differentiable at $x = a$, but not be twice differentiable at $x = a$. (Can you think of an example?) More generally, it is easy to construct functions $f : \mathbb{R} \to \mathbb{R}$ which are k-times differentiable at a point $x = a$, but not be $k + 1$-times differentiable. And it is also possible to construct functions $f : \mathbb{R} \to \mathbb{R}$ which are infinitely differentiable at $x = a$, but not be analytic at $x = a$. (These functions are not as obvious to construct, but we will investigate examples in a future class.)

How is the situation with complex-valued functions different? One of the main theorems of this class is that if a function $f : D \to \mathbb{C}$ is holomorphic at z, then it is analytic at z! That is, the existence of a single derivative at a point is enough to guarantee essentially the best regularity properties possible!

What does rigidity mean? Intuitively speaking, the rigidity of a holomorphic function refers to the remarkable fact that a holomorphic function $f : D \to \mathbb{C}$ is uniquely determined, or very highly determined, by simply knowing the values of f on a 'small' subset of D. Another theme of this class will be working out exactly what 'small' means, or what 'highly determined' means, and proving these facts.

Again, however, we can give a few examples which illustrate this phenomenon. For example, another of the main theorems we will prove will show that if $f : D \to \mathbb{C}$ is holomorphic throughout D, and C is a small circle whose center is at z, then the values of f on C completely determine the value not only of $f(z)$, but also the values of the derivatives $f^{(k)}(z)$ for all k. Again, this is entirely in contrast to the real valued case; it is easy to think of functions $f : \mathbb{R}^2 \to \mathbb{R}$ which are very smooth (say, analytic), but are still highly 'undetermined'.

2. Applications of complex analysis

We've seen a brief sampler of the very special properties holomorphic functions have. What are some applications of the results which we will study and prove? Well, the applications are numerous; really too numerous to comprehensively enumerate here. We will only consider a few of the more mathematical applications.

The regularity theorems we will study turn out to be extremely powerful tools that can be used to prove very deep theorems in mathematics. For example, halfway through the class we will be able to give a simple proof of the Fundamental Theorem of Algebra (this proof is certainly different than Gauss' original proof, since Gauss did not have any proofs of theorems in complex analysis when he found his first proof). We will also see the application of integration of holomorphic functions to problems which on the surface look purely like problems about functions of a real variable. For example, we will be able to calculate integrals which are essentially uncomputable without complex analysis (there is an interesting anecdote about these types of integrals in Richard Feynman's book *Surely You're Joking, Mr. Feynman!*).

Complex analysis turns out to be widely applicable to many other branches of mathematics, as well. For example, a huge portion of modern number theory is based on techniques from complex analysis. It is unlikely that we will have much time to explore these applications (although the textbook does contain them); for example, the first proof of the Prime Number Theorem, which provides asymptotics for the number of prime numbers less than X, as $X \to \infty$, almost exclusively uses complex-analytic techniques. Deeper investigations into the distribution of prime numbers naturally use complex analysis, as well.

Complex analysis also turns out to be closely related to *Fourier analysis*, which, loosely speaking, is the study of expanding functions $f : \mathbb{R} \to \mathbb{R}$ into infinite trigonometric sums. Fourier analysis is one of the most useful branches of mathematics in real-life, due to its usefulness in studying various physical phenomena (Fourier discovered many of the main results of the subject when trying to solve the partial differential equation which governs heat flow). This connection naturally leads to applications of complex analysis in physics and engineering. We will see some of the connection between complex analysis and partial differential equations when we see how holomorphic functions are closely related to *harmonic functions*, which are functions $f : \mathbb{R}^2 \to \mathbb{R}$ which satisfy a certain PDE.

Another important reason for studying complex analysis is that many of the ideas which come to dominate modern mathematics first had their origins in complex analysis. Many of the regularity results in PDEs were first inspired by the regularity results for harmonic functions which are natural consequences of regularity results for holomorphic functions. A substantial part of differential geometry was motivated by trying to understand the types of complex surfaces on which certain holomorphic functions could be defined. Many of the definitions in modern algebraic geometry (one of the greatest achievements of 20th century mathematics) are inspired by definitions and concepts which first appear in complex analysis.