MORE CONSEQUENCES OF CAUCHY'S THEOREM

Contents

We have stated Cauchy's Theorem and the Cauchy Integral Theorem, and already proven a remarkable number of corollaries of these two theorems. Among them are the regularity of holomorphic functions (holomorphic on an open set implies analytic on an open set), uniqueness theorems for holomorphic functions (analytic continuation, etc.), Liouville's Theorem (bounded functions are entire), and the Fundamental Theorem of Algebra (all non-constant complex polynomials have complex roots). We will gather some more selected consequences and applications of these ideas, many of which have been proven to be useful within complex analysis, in other branches of mathematics, or in various applications in science and engineering.

1. The Mean Value Property and the Maximum-Modulus Principle

We can use the Cauchy Integral Formula and the uniqueness of holomorphic functions to prove an interesting fact about holomorphic and harmonic functions.

Theorem 1 (The Mean Value Property)**.** *Let f be holomorphic on an open disc containing a circle C and its interior, and let C be centered at z_0 and have radius r. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

In other words, the value of $f(z_0)$ is the average (in the sense of mean) of the value of f along the circle C.

Proof. The Cauchy Integral Formula applied to the integral of f along C reads

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z_0}\, d\zeta.$$

Parameterize C with $z(t) = z_0 + re^{it}, 0 \leq t \leq 2\pi$. Then $z'(t) = ire^{it}$, and direct evaluation of the above integral yields

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{it}) \cdot ire^{it}}{re^{it}}\, dt = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

□

Corollary 1 (Mean Value Property for harmonic functions)**.** *Suppose u is harmonic on an open disc Ω, and C is a circle contained in Ω with center z_0 and radius r. Then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})\, dt.$$

(For simplicity of notation we identify $x + yi$ with the point $(x, y) \in \mathbb{R}^2$.)

Proof. One can show that if u is harmonic on an open disc, then it is the real part of a holomorphic function on that disc. (This is Exercise 12a in Chapter 2 of the text; the analogous statement for arbitrary open sets is false.) So let f be any holomorphic function with $\operatorname{Re} f = u$. Apply the Mean Value property to f and take real parts. □

Corollary 2 (The Maximum-Modulus Principle)**.** *Suppose f is holomorphic and non-constant on an open connected set containing a circle C and its interior. Let D be the closed disc enclosed by C. Then f takes its maximum value on D (in the sense of maximum absolute value) on the boundary circle C. In other words, f never has a relative maximum.*

Proof. Suppose f has an absolute maximum for D at a point z_0 in the interior of D. We want to derive a contradiction; in particular, we will show that f must be constant on D. Let C' be any circle centered at z_0 completely contained in D; such a circle exists since z_0 is in the interior of D. Then the mean value property implies

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt,$$

where r is the radius of C'. However, we also know that

$$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\, dt \le \max_{\zeta\in C'} |f(\zeta)|.$$

By assumption, $f(z_0)$ is an absolute maximum for D, and C' is contained in D, so we also have $|f(z_0)| \ge \max_{\zeta\in C'} |f(\zeta)|$; hence, we must actually have

$$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\, dt\right| = \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|\, dt = \max_{\zeta\in C'} |f(\zeta)|.$$

On the one hand, $|f(z_0)| \ge |f(z_0 + re^{it})|$ for all $0 \le t \le 2\pi$. On the other hand, we also have

$$0 = \int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{it})|\, dt,$$

by the above equalities. Since $|f(z_0)| - |f(z_0 + re^{it})|$ is a continuous function of t (since f is holomorphic on C' and $|\cdot|$ is continuous from $\mathbb{C}$ to $\mathbb{R}$), and the only way the integral of a non-negative continuous function over an interval $[a, b]$ is equal to 0 is if that function is equal to 0 on all of that interval, we must have $|f(z_0)| = |f(z_0 + re^{it})|$ for all $0 \le t \le 2\pi$; in other words, f has the same absolute value at z_0 as on any point on C'.

Since C' was any circle centered at z_0 contained in D, what we have actually shown is that f has constant absolute value on some open disc centered at z_0. (Namely, any open disc contained in D.) However, we showed in a homework assignment from about a month ago that any holomorphic function with constant absolute value on an open disc is actually constant on that disc. Therefore f is constant on some open disc; by uniqueness of analytic continuation f must be constant on any region containing this open disc.

□

Example. Consider a function like $f(z) = 1 - z^2$. When thought of as a function on $\mathbb{R}$, f obviously has a relative maximum $x = 0$. However, this is no longer a relative maximum in $\mathbb{C}$, because traveling in an imaginary direction from $z = 0$ (in particular, in a positive imaginary direction) increases the value of $f(z)$.

Example. Similarly, functions like $\cos z, \sin z$, which have relative maxima when thought of as real functions, no longer have those relative maxima as complex functions.

2. Morera's Theorem and some applications

Recall that Goursat's Theorem tells us the integral of a complex function holomorphic along a triangular contour of integration and its interior equals 0. Morera's Theorem provides a partial converse:

Theorem 2 (Morera's Theorem)**.** *Suppose f is a continuous complex function in an open disc Ω. Let T be any triangle contained in Ω. If $\int_T f(z)\,dz = 0$ for all such triangles, then f is holomorphic on Ω.*

Proof. This proof will be a little sketchy, and refers back to the proof of the theorem which states that a holomorphic function on an open disc has a primitive function. In that proof, we only used the fact that f was holomorphic in order to conclude that f was continuous and that $\int_T f(z)\,dz = 0$ on triangular contours. So we can reuse that proof to construct a primitive function $F(z)$ for $f(z)$ on Ω. In particular, F is holomorphic, and $F'(z) = f(z)$. But we also know that holomorphic functions are analytic, so this implies that $f(z)$ is holomorphic on Ω as well. □

Remarks.

- The Ω in the statement of the theorem can also be an open rectangle, triangle, keyhole contour, etc, and actually in general any connected region, such as an annulus. The reason regions like the annulus do not cause problems in the proof of Morera's Theorem is that being holomorphic at a point is a local property, so that to show f is holomorphic at a point z in Ω we really only need to know that integrals along small triangles near z equal 0. In particular, it is not necessary to know about the properties of f far away from z.
- Another alternate formulation of Morera's Theorem (essentially equivalent) replaces triangular contours with circular contours, rectangular contours, or even rectangular contours whose edges are parallel to the real and imaginary

axes. Also, Morera's Theorem is true even when restricted to just equilateral triangles (see Problem 3, Chapter 2 in the text.)

Why is Morera's Theorem useful? After all, given $f(z)$, it seems hard to check that integrals along every triangle equal 0. However, there are many practical situations where even though $f(z)$ is not described explicitly, one has information about other properties of f. The following corollary is a prime example.

Corollary 3. *Suppose $f_n(z)$ is a sequence of holomorphic functions defined on a region Ω, and $f_n(z)$ converges to $f(z)$ uniformly on every compact subset of Ω. (That is, on any compact $K \subset \Omega$, the functions $f_n(z)$ restricted to K converge uniformly to $f(z)$.) Then f is holomorphic on Ω.*

Proof. Let z be an arbitrary point of Ω, and let D be an open disc whose closure is contained in Ω. Then Cauchy's Theorem tells us that $\int_T f_n(z)\,dz = 0$ for any triangle T contained in D. On the other hand, we know that f_n converges uniformly to f in the closure of D, so

$$0 = \lim_{n\to\infty} \int_T f_n(z)\,dz = \int_T f(z)\,dz.$$

Since the f_n are continuous, f will also be continuous on D. Morera's Theorem applied to f on D tells us f is holomorphic on D. Since z was arbitrary, this means f is holomorphic on all of Ω. □

Remarks.

- If f_n converges uniformly to f on Ω, then it does so on every compact subset of Ω. However, as a practical matter, there are many situations where f_n does not converge uniformly to f on all of Ω, but only on compact subsets, and we would like to be able to apply this corollary to those situations.
- The analogue of this corollary for real differentiable functions (or even real analytic function) is false. For example, Weierstrass' approximation theorem says that any continuous function $f(x)$ on a closed interval $[a, b]$ can be approximated uniformly by polynomials. However, there are obviously continuous, non-differentiable functions, and polynomials are analytic, so certain sequences of polynomials uniformly converge to non-differentiable functions.
- Actually, we can go further than the above corollary, and it is possible to prove that not only is f holomorphic, but that the derivatives f_n' converge uniformly on compact subsets to f'. For a proof, see the textbook (we will skip it, although if you want you can use this property freely.)

  This is false in the real case: for example, the functions $f_n(x) = \sin(nx)/n$ converge uniformly to the differentiable function $f(x) = 0$, but $f_n'(x) = \cos(nx)$ do not converge to any function, let alone $f'(x) = 0$.

Another application of Morera's Theorem is related to a previous homework exercise.

Corollary 4. *Suppose f is continuous on an open set Ω and also holomorphic on all of Ω except possibly at one point (or more generally, a finite number of points). Then f is actually holomorphic on all of Ω.*

Proof. Let z be one of those points at which we do not yet know f is holomorphic. Let D be a small open disc containing z. By a previous homework assignment, we know that $\int_R f(z)\,dz = 0$ for any rectangle R in D where $z \notin R$. Actually, it is possible to modify the solution of that problem to let z be on R as well. Then, we can apply the rectangle version of Morera's Theorem to conclude that f is actually holomorphic on all of D, and in particular at z, as desired. □

This corollary will be useful in a few weeks, when we discuss the types of singularities a holomorphic function can have.

An important application of the result on uniform convergence of holomorphic functions on compact subsets is in analytic number theory:

Example. Let n be a positive integer, and let s be a complex number. Then we can define $n^s = e^{s\log n}$; notice that in this case log is well defined because n is a positive real number. (Why switch from the variable z to the variable s? Tradition!) Consider the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is called the *Riemann zeta function*. For positive real s, this is just the p-series from calculus, and the integral test for convergence tells us that $\zeta(s)$ converges when $s > 1$, and diverges for $s = 1$ (or $s < 1$).

We claim that $\zeta(s)$ converges for $\operatorname{Re} s > 1$, and that it is actually a holomorphic function on this entire half-plane. First, notice that each $1/n^s$ is holomorphic on $\operatorname{Re} s > 1$, because n^s is holomorphic and never equal to 0. Next, notice that $|n^s| = |e^{s\log n}| = e^{\operatorname{Re} s\log n} = e^{(\log n)(\operatorname{Re} s)}$. If we let $s = \sigma + it$ (why the weird letters? Tradition!), then we see that $|n^s| = e^{\sigma\log n} = n^\sigma$; in other words, $|n^s| = n^{\operatorname{Re} s}$. Therefore,

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^s}\right| = \sum_{n=1}^{\infty} \frac{1}{n^\sigma},$$

which converges as long as $\sigma > 1$. Therefore $\zeta(s)$ converges absolutely when $\sigma = \operatorname{Re} s > 1$, so in particular converges in this half-plane as well.

Finally, notice that the series defining $\zeta(s)$ converges uniformly on every compact subset contained in $\operatorname{Re} s > 1$. Indeed, if K is such a compact subset, then for every $s \in K$, actually $\operatorname{Re} s \geq 1 + \delta$ for some $\delta > 0$. (Short proof: the continuous function $f(s) = \operatorname{Re} s$ achieves its minimum on K, which must be > 1.) Therefore, we can apply the Weierstrass M-test with the bound $|1/n^s| \leq |1/n^{1+\delta}|$.

Why exactly this function is important in number theory is a fairly long story, but if you are interested in more you can read Chapters 6 and 7 of the textbook. The method described above is flexible enough to be used to prove that a wide class of complex functions in analytic number theory, known as *L-functions*, are holomorphic in right half-planes.

3. The Schwarz Reflection Principle

Consider the following fairly common question in analysis: given a continuous (or differentiable, or C^∞, or analytic) function f on a set Ω, is it possible to extend that function to a larger set while preserving continuity (or differentiability, etc.)? If it is possible, in how many different ways is it possible?

Obviously, for continuous functions there will be probably be many different ways to continually extend a function. (This certainly is true if f is continuous on a closed interval $[a, b]$ and we want to extend f to a larger closed interval, say).

We are interested in this question for holomorphic functions and certain sets Ω. In particular, we are interested in the case where Ω is some subset in the upper-half plane which intersects the real axis in a line segment. (For example, Ω might be the upper half of a disc centered on the real axis.) We are interested in extending this to a holomorphic function on the reflection of Ω across the real axis.

Before stating and proving the relevant theorem, let's look at some examples of holomorphic functions f and how $f(z)$ is related to $f(\overline{z})$. In each of these examples, $f(z)$ is real when z is real.

Examples.

- Let $f(z) = a_0 + \ldots + a_n z^n$ be any polynomial with real coefficients. Then $f(\overline{z}) = a_0 + \ldots + a_n \overline{z}^n$. Notice that $\overline{f(\overline{z})} = a_0 + \ldots + a_n z^n = f(z)$.
- Let $f(z) = 1/z$. Then $f(\overline{z}) = 1/\overline{z}$, and $\overline{f(\overline{z})} = 1/z = f(z)$.
- Let $f(z) = e^z$. If $z = x + yi$, then $f(\overline{z}) = e^{x-yi} = e^x e^{-yi}$. Notice that $\overline{e^x e^{-yi}} = e^x \overline{e^{-yi}} = e^x e^{yi} = e^z = f(z)$.

In every example we looked at, $\overline{f(\overline{z})} = f(z)$. This is not a coincidence, as the following theorem shows:

Theorem 3 (Schwarz Reflection Principle)**.** *Let Ω be an open connected set which is symmetric about the real axis. Let Ω^+ be the part of Ω consisting of points with positive imaginary part, Ω^- be the part with negative imaginary part, and I the boundary between Ω^+ and Ω^- on the real axis which lies in the interior of Ω. (In other words, I is the interior of the intersection of Ω with the real axis.)*

If f is a holomorphic function on Ω^+ which extends continuously to I in such a way so that f is real-valued on I, then f extends to a holomorphic function on all of Ω, with $f(\overline{z}) = \overline{f(z)}$ for $z \in \Omega$.

The proof will be an easy consequence of the following lemmas:

Lemma 1. *Suppose f is holomorphic on an open connected set Ω except possibly at the points of a line segment L (closed, open, or half-open) contained in Ω, where we only know that f is continuous. Then f is actually holomorphic on L as well.*

Proof. This is a generalization of the previous results on functions holomorphic everywhere except at a finite number of points. Without loss of generality, we may assume that the line segment L is parallel to the real axis. (If not, perform a rotation; it is easy to check the arguments we give will still be valid.) Furthermore, since f being holomorphic at a given $z \in L$ only depends on the values of f near z, we can take Ω to be an open disc (namely, any open disc containing z inside Ω.) We will use the version of Morera's Theorem where we want to check that $\int_R f(z)\,dz = 0$ for all rectangles R whose sides are parallel to the real and imaginary axes.

If R does not intersect or contain L, then $\int_R f(z)\,dz = 0$ by Cauchy-Goursat's Theorem. Suppose that part or all of one of the edges of R coincides with part or all of L. Then we can slightly shift R in a vertical direction to move R off L; this is possible without leaving Ω because R is compact so is some nonzero distance from the boundary of Ω. Let R_ε be this shift of R by ε in a vertical direction. Then $\int_{R_\varepsilon} f(z)\,dz = 0$ by Cauchy-Goursat. Also,

$$\lim_{\varepsilon \to 0^+} \int_{R_\varepsilon} f(z)\,dz = \int_R f(z)\,dz$$

by the continuity of f (this was essentially one of the previous homework problems), so actually $\int_R f(z)\,dz = 0$, as desired.

Finally, suppose that part or all of L is contained inside R. Then split R up into two rectangles, each of which has an edge intersecting L in a line segment, and apply the previous part to conclude $\int_R f(z)\,dz = 0$ as well.

In all possible cases, $\int_R f(z)\,dz = 0$, so by Morera's Theorem $f(z)$ is holomorphic on all of Ω, as desired. □

Lemma 2 (Symmetry Principle). *Suppose $\Omega, \Omega^+, I, \Omega^-$ are as in the statement of the Schwarz Reflection Principle. Let f^+ be a holomorphic function on Ω^+ and f^- be a holomorphic function on f^-. If f^+, f^- both continuously extend to I and agree on I, then the function $f(z)$ defined on Ω, with $f(z) = f^+(z)$ if $im z \geq 0$, $f(z) = f^-(z)$ if $im z \leq 0$, is holomorphic on Ω.*

Proof. Clearly f is holomorphic on Ω^+, Ω^-. The only possible points where f might not be holomorphic are points z on I. Let $z \in I$ be arbitrary, and let $D(z)$ be a small disc centered at z contained in Ω. Then f is holomorphic on $D(z)$ except possibly on the real axis part of $D(z)$. However, f is continuous there, so we can apply the previous lemma to conclude that f is actually holomorphic on all of $D(z)$, and in particular at z. Since z was arbitrary, this proves the lemma. □

Proof (of the Schwarz Reflection Principle). Let f be defined on all of Ω as in the statement of the Schwarz Reflection Principle. Clearly f is continuous; we only need to check that f is holomorphic on I and Ω^-. By the previous lemmas it is sufficient to check that f is holomorphic on Ω^-. (Notice we need to know that f is real-valued on I for the functions $f(z), \overline{f(\overline{z})}$ to be equal on the real axis.)

Let $z_0 \in \Omega^+$ be arbitrary; we want to show f is holomorphic at $\overline{z_0}$. Since f is holomorphic at z, we can find a power series expansion $g(z) = \sum a_n (z - z_0)^n$ centered at z_0 valid in some open disc, where $f(z) = g(z)$. But then $f(\overline{z}) = \overline{f(z)} = \overline{g(z)} = \sum \overline{a_n}(\overline{z} - \overline{z_0})^n$ gives a valid power series expansion for f centered at $\overline{z_0}$ in some open disc; hence f is also holomorphic at $\overline{z_0}$, as desired. □