SINGULARITIES AND ZEROS OF HOLOMORPHIC FUNCTIONS

Contents

So far, our focus of study has been holomorphic functions. We will now concentrate on understanding points where functions are not holomorphic. In particular, we want to generalize our understanding of the behavior of $f(z) = 1/z$ near $z = 0$ to a broader range of functions, and eventually prove interesting theorems about those functions. In particular, we will prove a generalization of Cauchy's Theorem to functions which are more general than holomorphic functions.

1. Singularities: an introduction

Let f be a function defined on an open set Ω. If f is not defined at z_0, but is defined in some *punctured disc* (sometimes also called a *deleted neighborhood*) $0 < |z - z_0| < r$, then we call z_0 a *(point, isolated) singularity* of f. The following three examples of singularities turn out to represent each of the three broad types of behavior of singularities in complex analysis:

Examples.

- Consider $f(z) = 1/z$ defined on $\mathbb{C} - 0$. Then $z = 0$ is a singularity of f, since f is undefined at 0, but defined everywhere else. Notice that $|f(z)| \to \infty$ as $z \to 0$, regardless of the direction z approaches 0 in. More generally, $f(z) = 1/z^n$, where $n \geq 1$ is a positive integer, has $z = 0$ as a singularity, and $|f(z)| \to \infty$ as $z \to 0$. Recall that $\int_{S^1} 1/z^n \, dz = 2\pi i$ if $n = 1$ and 0 otherwise.
- Consider $f(z) = z/z$ defined on $\mathbb{C} - 0$. This is just the constant function $f(z) = 1$ defined on $\mathbb{C} - 0$. With this definition, $z = 0$ is a singularity of $f(z)$. However, notice that we can just define $f(0) = 1$ to make $f(z)$ not just continuous, but actually holomorphic at $z = 0$. (Recall that if we can define $f(0)$ to make f continuous at 0, f will automatically be holomorphic at 0, because f is already holomorphic in a punctured disc centered at 0.) Furthermore, notice that, unlike the previous example, $|f(z)|$ is bounded near $z = 0$.
- Consider $f(z) = e^{1/z}$. This example is genuinely different from the previous two examples. On the one hand, it is impossible to define $f(0)$ to make f continuous at 0, because as you saw in a homework assignment, $e^{1/z}$ takes every nonzero value infinitely often near 0. On the other hand, $|f(z)| \not\to \infty$ as $z \to 0$ as well.

If z_0 is a singularity of $f(z)$ and it is possible to define $f(z_0)$ in such a way to make f holomorphic at z_0, then we call z_0 a *removable singularity* of f. In some ways, removable singularities are the least interesting type of singularity, because they arise from holomorphic functions with a few points in the domain deleted. Nevertheless one can prove interesting theorems about removable singularities, which we will do later.

2. Zeros of holomorphic functions

As the example $f(z) = 1/z^n$ might indicate, a good strategy for understanding singularities might be to understand zeros of holomorphic functions first, since the singularities of $1/z^n$ arise at the points where the denominator z^n has zeros.

The main theorem on zeros which we will use is the following:

Theorem 1. *Suppose $f(z)$ is a holomorphic function on an open set Ω which is not identically zero. Let $z_0 \in \Omega$ be a point with $f(z_0) = 0$. Then there exists a unique positive integer n such that there exists an open disc U containing z_0 and a holomorphic function $g(z)$ defined on U such that $g(z)$ is nonzero on U and $f(z) = (z - z_0)^n g(z)$ on U.*

Proof. Because f is holomorphic at z_0, we can find a power series expansion for f at z_0:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

The assumption that f is not identically 0 guarantees that not all coefficients a_k equal 0. (If all the coefficients did equal 0, then $f = 0$ on some open disc, which implies f is 0 on all of Ω.) Furthermore, since $f(z_0) = 0$, $a_0 = 0$. Let n be the smallest positive integer such that $a_n = 0$. We claim that n is the integer in the theorem.

In the disc of convergence of the above power series, we have

$$f(z) = (z - z_0)^n \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}.$$

Notice that after factoring out $(z - z_0)^n$ we have another power series which converges in the same disc as the original power series. Let $g(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^{k-n}$ be this new power series; evidently g is holomorphic in its open disc of convergence. We need to show that there exists some open disc U containing z_0 such that $g(z)$ is nonzero on all of U.

Since $a_n \neq 0$, we must have that $g(z_0) \neq 0$. Since $g(z)$ is continuous on its domain, there exists some open disc centered at z_0 such that g is never 0 on that open disc. Take U to be this open disc.

Finally, we need to prove that the n described above is unique. Suppose m were another positive integer such that there existed an open disc V and function $h(z)$ such that $f(z) = (z - z_0)^m h(z)$ on V. Then we have $f(z) = (z - z_0)^n g(z) = (z - z_0)^m h(z)$ on $U \cap V$, which is an open disc containing z_0. Without loss of generality we may assume $n \geq m$; then $(z - z_0)^{n-m} g(z) = h(z)$ at all points except z_0. If $n > m$, then $(z - z_0)^{n-m} g(z) \to 0$ as $z \to z_0$, which by continuity of h at z_0 implies that $h(z_0) = 0$, contradicting the assumption that $h(z_0) \neq 0$. Therefore $m = n$, as desired. □

The unique positive integer n in the above theorem is called the *order* of the zero of f at z_0. A zero of order 1 is often called a simple zero.

Examples.

- $f(z) = z^n$ has a zero of order n at $z = 0$. Indeed, $z^n = (z^n) \cdot 1$.
- $f(z) = \sin z$ has a zero of order 1 at $z = 0$. To see this, use the power series expansion for $\sin z$ at $z = 0$:

$$\sin z = z - \frac{z^3}{3!} + \dots.$$

  By the proof of the previous theorem, the order of the zero $z = 0$ is the power in the first nonzero term of the power series expansion, hence is $z = 1$. By periodicity of $\sin z$, each of the zeros $z = n\pi$ is also a zero of order 1.
- Suppose $f(z)$ has a zero of order n at z_0 and $g(z)$ has a zero of order m at z_0. Then $fg(z)$ (the product of f and g) has a zero of order $n + m$ at z_0. Indeed, if we can write $f(z) = (z - z_0)^n h_1(z)$ and $g(z) = (z - z_0)^m h_2(z)$ near z_0, where $h_1(z_0), h_2(z_0) \neq 0$, then $fg(z) = (z - z_0)^{n+m} h_1(z) h_2(z)$ near z_0, and $h_1(z_0) h_2(z_0) \neq 0$, so $h_1 h_2$ is nonzero in some open disc containing z_0.

3. Poles of holomorphic functions

Suppose f is holomorphic and has a singularity at z_0. If f is nonzero in some punctured disc centered at z_0, then $1/f$ is holomorphic in that punctured disc. If we define $1/f$ at z_0 to be equal to 0, we say that f has a *pole of order* n if $1/f$ is holomorphic at z_0 and has a zero of order n at z_0. If z_0 is a pole of order 1 for f, we sometimes say that z_0 is a *simple pole* of f. Alternately, f has a pole at z_0 if $1/f$ is nonzero in some punctured disc centered at z_0 and $\lim_{z \to z_0} 1/f(z) = 0$.

Examples.

- The simplest and most useful example of poles are the functions $f(z) = 1/z^n$, where $n \geq 1$ is a positive integer. Then $1/f = z^n$ in all of $\mathbb{C} - 0$, and if we define $1/f(0) = 0$, then $1/f$ is holomorphic at 0. Since we already know that z^n has a zero of order n at 0, this means that $1/z^n$ has a pole of order n at $z = 0$.
- Consider the function $f(z) = 1/z + 1/(z - 1)$. This is holomorphic on all of $\mathbb{C} - \{0, 1\}$. We have

$$\frac{1}{f} = \frac{1}{\frac{1}{z} + \frac{1}{z-1}} = \frac{z(z-1)}{z - 1 + z} = \frac{z(z-1)}{2z - 1}.$$

  Notice that $\lim_{z \to 0} 1/f = 0(-1)/(-1) = 0$, which means that $1/f$ is continuous at $z = 0$ if we define $1/f(0) = 0$. Since $1/f$ is already holomorphic in a punctured disc centered at $z = 0$, this means that $1/f$ is also holomorphic at $z = 0$. Also, we can see that $1/f$ has a zero of order 1 at $z = 0$, since we can write $1/f = z((z - 1)/(2z - 1))$, where $(z - 1)/(2z - 1)$ is holomorphic in a neighborhood of 0 and is nonzero in that neighborhood. Therefore $z = 0$ is a pole of order 1 for f. Similarly, $z = 1$ is also a pole of order 1.

- Consider $f(z) = 1/(\sin z)$. Then $1/f = \sin z$ for $z \neq n\pi$, and if we define $1/f(n\pi) = 0$, then $1/f$ is continuous, hence holomorphic, at $z = n\pi$. We already showed that $\sin z$ had zeros of order 1 at $z = n\pi$, so $1/(\sin z)$ has poles of order 1 at $z = n\pi$.

The following propositions state basic facts about poles which we will frequently use, either implicitly or explicitly.

Proposition 1. *Suppose f has a pole of order n at z_0. Then there exists an open disc D centered at z_0 and a nonzero holomorphic function $h(z)$ on D such that $f(z) = (z - z_0)^{-n}h(z)$ on the punctured disc $D - \{z_0\}$.*

Proof. Since $1/f$ is holomorphic at z_0 and has a zero of order n there, we may write $1/f(z) = (z - z_0)^n g(z)$ in an open disc D centered at z_0 such that $g(z) \neq 0$ on D. Then $f(z) = (z - z_0)^{-n}/g(z)$, so set $h(z) = 1/g(z)$. Since $g(z)$ is never zero on D, this means $h(z)$ is nonzero and holomorphic on D as well. □

Proposition 2. *Suppose f has a pole of order n at z_0. Then there exists an open disc D centered at z_0, a holomorphic function $G(z)$ on D, and complex numbers $a_{-n}, \ldots, a_{-1}$ such that*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z - z_0} + G(z)$$

in $D - \{z_0\}$. Furthermore, $a_{-n} \neq 0$.

Proof. Using the previous proposition, there exists an open disc D containing z_0 such that $f(z) = (z - z_0)^{-n}h(z)$ on D, where $h(z)$ is nonzero on D. In particular, we can find a power series expansion $\sum b_k(z - z_0)^k$ for $h(z)$ centered at z_0, where $b_0 \neq 0$. If necessary we shrink D so that this power series converges on all of D. Then

$$f(z) = (z - z_0)^{-n}h(z) = \sum_{k=0}^{\infty} b_k(z - z_0)^{-n+k}.$$

We can then take $a_{-n} = b_0, a_{-(n-1)} = b_1, \ldots, a_{-1} = b_{n-1}$, and $G(z) = \sum_{k=n}^{\infty} b_k(z - z_0)^{k-n}$. □

Both of these propositions have converses, in the sense that any function of the types described above (such as $(z - z_0)^{-n}h(z), h(z_0) \neq 0$) are functions with poles of order n at z_0. The non-holomorphic part of the above expression; that is, the function

$$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z - z_0},$$

is called the *principal part* of $f(z)$ at z_0. Also, the complex number a_{-1} (we have $n \geq 1$, so there will always be a number a_{-1}) is called the *residue* of f at z_0. We sometimes write $\text{res}_{z=z_0} f(z)$ for this complex number.

The reason we single out a_{-1} for special mention is because it belongs to the only part of the principal part of $f(z)$ at z_0 which has no primitive function in any punctured disc centered at z_0. In particular, notice that for any circle C centered at z_0 with positive orientation,

$$\int_C \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z-z_0}\, dz = 2\pi i a_{-1}.$$

More generally, if C is sufficiently small (small enough to fit in the disc D in the previous proposition), then

$$\int_C f(z)\, dz = 2\pi i a_{-1},$$

since $\int_C G(z)\, dz = 0$ by Cauchy's Theorem.

Examples.

- The function $f(z) = 1/z$ has a simple pole at $z = 0$, and the residue of that pole is 1.
- If $n \geq 2$ is a positive integer, then $f(z) = 1/z^n$ has a pole of order n at $z = 0$, and the residue of that pole is 0. Notice that $\int_C 1/z^n\, dz = 0$ for any circle centered at 0.
- If $f(z) = 1/(z+i)^2 + 3/(z+i)$, then $f(z)$ has a pole of order 2 at $z = -i$, and this pole has residue 3. Given any circle C centered at $-i$, $\int_C f(z)\, dz = 3 \cdot 2\pi i = 6\pi i$.
- If $f(z) = \dfrac{\sin z}{z^2}$, then $z = 0$ is a pole of order 1. Indeed,

  $$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \ldots,$$

  so we see that $\sin z/z^2$ has a pole of order 1 and residue 1 at that pole.

Sometimes it is either computationally intensive or simply not feasible to directly compute the principal part of a pole of $f(z)$. In certain situations, it is still possible to calculate the residue of poles of $f(z)$; for example, the following proposition is sometimes useful.

Proposition 3. *Suppose $f(z)$ has a simple pole at z_0. Then*

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z).$$

More generally, if $f(z)$ has a pole of order n at z_0, then

$$\operatorname{res}_{z=z_0} f(z) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z).$$

Proof. By the previous proposition, in an open disc D containing z_0 we have

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z-z_0} + G(z),$$

so

$$(z-z_0)^n f(z) = a_{-n} + a_{-(n-1)}(z-z_0) + \ldots + a_{-1}(z-z_0)^{n-1} + G(z)(z-z_0)^n.$$

If we differentiate this $n-1$ times with respect to z, we get

$$\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z) = (n-1)!a_{-1} + (z-z_0)H(z),$$

for some holomorphic function $H(z)$ in D. (Apply the product rule repeatedly to see this is true.) Then the proposition follows by taking a limit as $z \to z_0$. □

The previous examples are useful in sometimes useful in situations where calculating the principal part exactly is difficult, but we have information about power series of certain relevant functions.

Examples.

- Let $f(z) = 1/\sin z$. Suppose we want to compute the residue of the simple pole $z = 0$. Then the previous proposition implies that
$$\text{res}_{z=0} 1/\sin z = \lim_{z\to 0} \frac{z}{\sin z}.$$
We know that $\lim_{z\to 0} \sin z/z = 1$ (for example, use the power series expansion for $\sin z$), so the residue of $z = 0$ for $1/\sin z$ is 1.
Similarly, consider $f(z) = 1/\sin(\pi z)$. This function has simple poles at all integers. At the integer $z = 0$, the residue is given by $\lim_{z\to 0} z/\sin \pi z$. By similar reasoning as before, $\lim_{z\to 0} \sin \pi z/z = \pi$, so the residue of $1/\sin(\pi z)$ at $z = 0$ is $1/\pi$.
- Let $f(z) = (z^2 + 2z - i)/(z + 2)$. Then $z = -2$ is a simple pole of $f(z)$. (This is true because the numerator, $z^2 + 2z - i$, is not 0 at $z = -2$.) Then
$$\text{res}_{z=-2} f(z) = \lim_{z\to -2}(z+2)f(z) = \lim_{z\to -2} z^2 + 2z - i = -i.$$
- Let $f(z) = 1/(z-1)^2 + z^2/(z-1)$. Then $z = 1$ is a pole of order 2, because $\lim_{z\to 1}(z-1)^2 f(z) = \lim_{z\to 1} 1 + z^2(z-1) = 1 \neq 0$, while $\lim_{z\to 0}(z-1)^3 f(z) = \lim_{z\to 1}(z-1) + z^2(z-1)^2 = 0$.
To compute the residue at $z = 1$, we use
$$\text{res}_{z=1} f(z) = \lim_{z\to 1} \frac{1}{1!}\frac{d}{dz}(z-1)^2 f(z) = \lim_{z\to 1} \frac{d}{dz} 1 + z^2(z-1) = \lim_{z\to 1} 3z^2 - 2z = 1.$$
- If you need to find the residue of $f(z)/z^n$, or more generally $f(z)/(z-z_0)^n$, and you know how to compute the power series expansion of $f(z)$ centered at z_0, you can use this information to compute residues. For example, let n be a positive integer. Then e^z/z^n has a pole of order n at $z = 0$, and we have an expansion
$$\frac{e^z}{z^n} = \frac{1}{z^n} + \frac{1}{z^(n-1)} + \frac{1}{2!z^{n-2}} + \ldots + \frac{1}{(n-1)!z} + \ldots,$$
where the rest of the terms correspond to an entire function. From this we can directly read off the residue of $1/(n-1)!$ for the pole $z = 0$.

In practice, poles frequently arise by dividing holomorphic functions by other holomorphic functions, and looking at points where the denominator has a zero of higher order than the numerator. However, there are examples of functions which cannot obviously be written as the quotient of holomorphic functions yet still have poles; for example, the Riemann zeta function $\zeta(s)$ is such an example; Riemann proved this has a pole of order 1 at $s = 1$. (Although we know that $\lim_{s\to 1^+} \zeta(s) = \infty$, this by itself does not rule out the possibility that $s = 1$ is not a pole. More work is required to check that $|\zeta(s)| \to \infty$ as $s \to 1$, in any direction.)

4. Distinguishing poles from removable singularities

Are there any properties that uniquely characterize removable singularities or poles? It turns out that the answer is yes.

Theorem 2. *(Riemann's Theorem on Removable Singularities) Suppose z_0 is a singularity of f, and $\lim_{z\to z_0}(z - z_0)f(z) = 0$. Then z_0 is a removable singularity of f.*

Proof. Define a new function $g(z) = (z - z_0)f(z)$ if $z \neq z_0$, and $g(z_0) = 0$. Then $g(z)$ is continuous at z_0 by assumption. Also, g is holomorphic on a punctured disc centered at z_0, so g is also holomorphic at z_0. Now let $h(z) = \dfrac{g(z) - g(z_0)}{z - z_0}$ for $z \neq z_0$, and $g'(z_0)$ otherwise. Then h is holomorphic at z_0 and also near z_0, and also when $z \neq z_0$, $h(z) = (z - z_0)f(z)/(z - z_0) = f(z)$. Therefore $h(z)$ is just $f(z)$ with $h(z_0)$ defined in such a way to make $f(z)$ holomorphic at z_0, so z_0 is a removable singularity of z_0. □

Corollary 1. *z_0 is a removable singularity of f if and only if f is bounded on some punctured disc centered at z_0.*

Proof. If z_0 is a removable singularity, then defining $f(z_0)$ appropriately, we find that f is continuous at z_0, so on a sufficiently small punctured disc f will be bounded.

Conversely, if f is bounded as $z \to z_0$, apply the previous theorem to see that z_0 is a removable singularity. □

In contrast to removable singularities, a pole z_0 is characterized by $f(z) \to \infty$ as $z \to z_0$. As a matter of fact, some sources take this as the definition of a pole (versus our definition of a pole as a point z_0 where $1/f$ has a zero of some order.)

Theorem 3. *If z_0 is a pole of f, then $|f(z)| \to \infty$ as $z \to \infty$.*

Proof. By a previous theorem, we know that near z_0, we can write $f(z) = (z - z_0)^{-n}h(z)$, for some holomorphic function h (recall h is holomorphic on not just some punctured disc centered at z_0, but an actual disc centered at z_0). Since h is bounded at and near z_0, $\lim_{z\to z_0} |f(z)| = \lim_{z\to z_0} |(z - z_0)^{-n}h(z)| = \infty$. □

As a matter of fact, our theorem on the local structure of functions near poles tells us that near a pole, a holomorphic function tends to infinity as an integer power of $z - z_0$, because if n is the order of a pole z_0 for f, then

$$\lim_{z\to z_0} (z - z_0)^n f(z) = \lim_{z\to z_0} h(z) = h(z_0) \neq 0.$$

Examples.

- Suppose you are told that $f(z)$ is a holomorphic function on $\mathbb{C} - 0$ satisfying $|f(z)| \leq |z|^{-1/2}$ for all nonzero z (or all nonzero z close to 0). Then $\lim_{z\to 0} zf(z) = 0$, since $|zf(z)| \leq |z|^{1/2}$, which tends to 0 as $|z| \to 0$. By Riemann's principle of removable singularities, this implies that $f(z)$ actually has a removable discontinuity at 0, so is actually bounded near 0.
- As a matter of fact, this shows (via a roundabout way) that it is impossible to define a holomorphic extension of $f(x) = \sqrt{x}$ from the positive real line to all of $\mathbb{C}$; if this were possible, then $1/\sqrt{z}$ would be holomorphic on $\mathbb{C} - 0$ and would satisfy $|f(z)| \leq |z|^{-1/2}$, implying that $1/\sqrt{z}$ has a removable singularity at 0. But this is impossible, since $|\sqrt{z}|$ should approach 0 as $z \to 0$, so that $|1/\sqrt{z}| \to \infty$ as $z \to 0$, a contradiction.

So, in summary, we see that removable singularities of f are characterized by f being bounded near that singularity, while poles of f are characterized by $|f|$ tending to infinity near that pole. Any singularity which does not fall under either of these two possibilities is called an *essential singularity.* In many respects, these are the hardest singularities to understand.

Example. $f(z) = e^{1/z}$ has $z = 0$ as an essential singularity. Indeed, a homework assignment showed that $f(z)$ is not bounded near 0, nor is $|f(z)|$ approaching infinity as $z \to 0$. On the contrary, we saw that $e^{1/z}$ takes every nonzero value infinitely often in any punctured disc (no matter how small) centered at 0!

The following theorem gives partial information on the values that a general holomorphic function takes near an essential singularity.

Theorem 4 (Casorati-Weierstrass)**.** *Suppose z_0 is an essential singularity of f. Let $D - \{z_0\}$ be any punctured disc centered at z_0 contained in the domain of f. Then the image of this punctured disc under f is dense in the complex plane. (In other words, given any $\varepsilon > 0$ and $w \in \mathbb{C}$, there exists some $z \in D - \{z_0\}$ such that $|f(z) - w| < \varepsilon$.)*

Proof. We prove the theorem by contradiction. Suppose that there existed $w \in \mathbb{C}$ and some $\varepsilon > 0$ such that $|f(z) - w| > \varepsilon$ for all $z \in D - \{z_0\}$. Consider the function $g(z) = 1/(f(z) - w)$; since $f(z) \neq w$ on $D - \{z_0\}$ this function is holomorphic on this disc. Also, because $|f(z) - w| > \varepsilon$, this means that $g(z) < 1/\varepsilon$ on all of $D - \{z_0\}$. By the previous corollary on removable singularities, this implies that z_0 is a removable singularity of $g(z)$. Define $g(z_0)$ appropriately to make $g(z)$ holomorphic at z_0.

There are two possibilities: if $g(z_0) \neq 0$, then

$$\lim_{z\to z_0} \frac{1}{f(z) - w} = g(z_0) \Rightarrow \lim_{z\to z_0} f(z) = w + \frac{1}{g(z_0)},$$

contradicting the hypothesis that z_0 was an essential singularity (hence not removable) of f.

The other possibility is that $g(z_0) = 0$. But then this means that $\lim_{z\to z_0)} g(z) = 0$, or, in other words, $\lim_{z\to z_0} |f(z) - w| = \infty$, which implies that z_0 is a pole of f, again contradicting our assumption. □

Actually, this theorem does not tell the whole truth about essential singularities. Something much stronger is true:

Theorem 5 (Picard's Big Theorem)**.** *Suppose* z_0 *is an essential singularity of* f*. If* $D - \{z_0\}$ *is any punctured disc centered at* z_0 *contained in the domain of* f*, then the image of this punctured disc under* f *is all of* $\mathbb{C}$ *except possibly one point. Furthermore, each value in the image is achieved infinitely often.*

We do not provide the proof here, as it requires slightly more advanced techniques. In any case, notice that Picard's (Big) Theorem says that the phenomenon we observed with $e^{1/z}$ is true for any function with an essential singularity.