THE RESIDUE THEOREM

Contents

Now that we have discussed basic properties of poles and removable singularities, we will prove a generalization of Cauchy's Theorem which can handle a functions with a finite number of poles, and discuss a few applications.

1. The Residue Formula

Suppose f has z_0 as a pole but is holomorphic everywhere else. We would like to evaluate integrals of the form $\int_C f(z)\,dz$, where C is a circle which encloses z_0. For example, if C is a unit circle and $f(z) = 1/z$, we are in this situation. The following theorem and its obvious generalizations provide the key idea:

Theorem 1 (The Residue Formula). *Suppose f is holomorphic on an open set containing a circle C (with positive orientation) and its interior, except at a point z_0 inside C where f has a pole. Then*

$$\int_C f(z)\,dz = 2\pi i \operatorname{res}_{z=z_0} f(z).$$

Proof. The proof is actually fairly easy given what we have proven already. First, recall that in the proof of the Cauchy Integral Formula we made use of a keyhole contour which enclosed z_0, and let the corridor width go to 0. Using the same idea here, we find that

$$\int_C f(z)\,dz = \int_{C_\varepsilon} f(z)\,dz,$$

where C_ε is a small circle of radius ε centered at z_0. We also proved earlier that inside some (sufficiently small) disc centered at z_0, we could write

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z-z_0} + G(z),$$

where $G(z)$ is holomorphic. If we choose ε so that C_ε is small enough so that this expression is valid on all of C_ε and its interior, then we have

$$\int_C f(z)\,dz = \int_{C_\varepsilon} f(z)\,dz = \int_{C_\varepsilon} \frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-(n-1)}}{(z-z_0)^{n-1}} + \ldots + \frac{a_{-1}}{z-z_0} + G(z)\,dz.$$

We can integrate this expression term-by-term; notice that all but one of these terms have primitives in the punctured disc enclosed by C_ε. (For the $a_{-k}/(z-z_0)^k$ terms, for $k \geq 2$, we have an explicit primitive, and $G(z)$ has a primitive because it is holomorphic in all of C_ε and its interior, including the point z_0.) In particular the above expression actually equals

$$\int_{C_\varepsilon} \frac{a_{-1}}{z-z_0}\, dz = 2\pi i a_{-1} = 2\pi i \operatorname{res}_{z=z_0} f(z),$$

as desired. □

This theorem has obvious generalizations to integrals over contours C which are not necessarily circles; since we are using the idea in the proof of the Cauchy Integral Formula we can use any contour which is permissible in the Cauchy Integral Formula. At this point for us, this includes contours like rectangles, triangles, ellipses, keyhole contours, etc. There is also the obvious generalization to the case where f has finitely many poles inside C.

Corollary 1. *Suppose f is holomorphic on a curve C as above (circle, triangle, rectangle, etc.) and its interior except at a finite number of poles $z_1, z_2, \ldots, z_n$ in the interior of C. Then*

$$\int_C f(z)\, dz = 2\pi i(\operatorname{res}_{z=z_1} f(z) + \ldots + \operatorname{res}_{z=z_n} f(z)).$$

Examples.

- This theorem makes a variety of calculations which we did in the past fairly trivial. For example, recall a homework assignment which asked you to show that $\int_C 1/(z-a)\, dz = 2\pi i$ if C was a circle containing the point a. The function $f(z) = 1/(z-a)$ has a pole at $z = a$ of order 1 and residue 1, so the residue theorem immediately implies that $\int_C 1/(z-a)\, dz = 2\pi i$.
- The residue formula is powerful enough to allow flexibility in the shape of C. For example, our calculation that $\int_C 1/z\, dz = 2\pi i$ for circles centered at 0 used an explicit parameterization of C, and the calculation was easy because of cancellation in terms. Direct evaluation of $\int_\gamma 1/z\, dz$ where γ is a rectangular contour, triangle, ellipse, etc. containing would be much more tedious (though still possible), but the residue theorem immediately tells us these are also all equal to $2\pi i$ as well.
- Similarly, recall the homework assignment which asked for the calculation of $\int_C 1/(z-a)(z-b)\, dz$, where C was a circle containing a but not b. The function $1/(z-a)(z-b)$ has poles of order 1 at $z = a$ and $z = b$, and the residues at those points are $1/(a-b)$ and $1/(b-a)$, respectively. Therefore the residue formula implies that $\int_C 1/(z-a)(z-b)\, dz = 2\pi i/(a-b)$, which was the answer you did a non-trivial calculation to compute.

  If C had contained both a, b, then the residue theorem implies that $\int_C 1/(z-a)(z-b)\, dz = 2\pi i/(a-b) + 2\pi i/(b-a) = 0$.

2. Applications and corollaries of the residue formula

The residue formula has some easy corollaries which nevertheless are useful in a variety of situations. Once such corollary provides a way (at least in principle) to obtain information on the number of poles and zeros of a function inside a contour.

Suppose $f(z)$ is holomorphic on an open set Ω except at a finite number of poles $z_1, \ldots, z_n$. Consider the function $f'(z)/f(z)$: this is holomorphic everywhere on Ω except at the poles z_i as well as the zeros of $f(z)$. Suppose w_i is a zero of order n_i of $f(z)$. Then near w_i we may write

$$f(z) = (z - w_i)^{n_i} h_i(z),$$

where $h_i(z)$ is holomorphic and nonzero near w_i. Then $f'(z) = n_i(z - w_i)^{n_i - 1} h_i(z) + (z - w_i)^{n_i} h_i'(z)$, so

$$\frac{f'(z)}{f(z)} = \frac{n_i(z - w_i)^{n_i} h_i(z)}{(z - w_i)^{n_i} h_i(z)} + \frac{(z - w_i)^{n_i} h_i'(z)}{(z - w_i)^{n_i} h_i(z)} = \frac{n_i}{(z - w_i)} + \frac{h_i'(z)}{h_i(z)}.$$

Notice that h_i'/h_i is holomorphic at and near w_i, since h_i is nonzero at and near w_i. Therefore f'/f has a pole of order 1, with residue n_i at w_i.

Similarly, if z_i is a pole of order m_i, one can compute that f'/f has a pole of order 1 with residue $-m_i$ at m_i. (This is a good exercise to check on your own.) As we mentioned earlier, the only singularities of f'/f occur at either poles of f or zeros of f. Therefore, we an application of the residue formula proves the following theorem:

Theorem 2 (Argument Principle)**.** *Suppose $f(z)$ is holomorphic in an open set Ω except at a finite number of poles. Let γ be any circle (or more generally, any contour permissible in the residue formula) on which f has no zeros or poles. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz = N - P,$$

where N is the total number of zeros of f inside γ and P the total number of poles inside γ (each counted with multiplicity).

Why is this result sometimes called the argument principle? Consider the following heuristic (ie, non-rigorous) reason: we can think of f'/f as being the derivative of $\log f(z)$ (if it is possible to define such a function). On the other hand, $\log re^{it} = \log r + it$, where t is ambiguous up to a multiple of $2\pi i$. Therefore, we can think of $\int_\gamma f'/f$ as measuring the total change in the argument of f as we travel along γ. This can be made rigorous, but is probably best illustrated with a few examples.

Examples.

- Let $f(z) = z$ and γ be the unit circle. As we wind our way around the unit circle, the argument of $f(z)$ changes from 0 (say) to 2π. This is reflected by the fact that $f(z) = z$ has exactly one zero of order 1 inside γ, so that $N = 1, P = 0$.
- Let $f(z) = z^2$ and γ be the unit circle. As we wind our way around the unit circle, the argument of $f(z)$ changes from 0 (say) to 4π. Indeed, letting $z(t) = e^{it}$, we see that $f(z(t)) = e^{2it}$, so the argument changes from 0 to 4π.

This is reflected in the fact that $f(z) = z^2$ has a single zero of order 2 inside γ, so that $N = 2, P = 0$.
- Let $f(z) = 1/z$ and γ be the unit circle. As we wind our way around the unit circle, the argument of $f(z)$ changes from 0 (say) to -2π. This is reflected by the fact that $f(z) = 1/z$ has exactly a single pole of order 1 inside γ, so that $N = 0, P = -1$.

The argument principle actually has useful and important numerical applications, as well. Suppose you know that $f(z)$ is holomorphic, say, so has no poles, and you want to compute the zeros of $f(z)$, or at least you know the location of all the poles of $f(z)$. For example, this is the case for the Riemann zeta function. If you want to locate the zeros of $f(z)$, you can try evaluating $\int_\gamma f'/f$ for various contours γ numerically, even if you do not actually know how to integrate f'/f. Because $N - P$ is always an integer, you as you make better numerical approximations to $\int_\gamma f'/f$, your answers will converge towards an integer.

If you can obtain reasonable bounds on the error of your numerical integration, then by taking a sufficiently accurate numerical approximation of the exact value of the integral you can determine the nearest integer to your numerical calculation, and with good error bounds and sufficiently accurate approximation this will actually compute $N - P$ correctly.

The following theorem sometimes provides a way to count the number of zeros of a holomorphic function inside a contour:

Theorem 3 (Rouche's Theorem)**.** *Suppose f and g are holomorphic functions in and on a closed curve C which has an interior (so, a circle, polygon, ellipse, etc.). If $|f| > |g|$ on C, then f and $f + g$ have the same number of zeros inside C. (Notice that $|f| > |g|$ automatically implies that f, $f + g$ are nonzero on C.)*

Proof. Let $0 \leq t \leq 1$. Consider the function $f_t(z) = f(z) + tg(z)$. We can think of this as a continuous deformation of the function $f(z)$ to the function $f + g$ of z. If n_t is the number of zeros of f_t inside of C, then to prove the theorem we will show that n_t is a continuous function of t. Since n_t is an integer, if we prove this n_t must be constant for $0 \leq t \leq 1$, so that $f, f + g$ have the same number of zeros inside t. (If n_t is continuous, then n_t cannot change, for that would be a discontinuity in n_t.)

Since f_t has no zeros or poles on C, the argument principle tells us that

$$n_t = \int_C \frac{f_t'(z)}{f_t(z)}\,dz.$$

On the other hand, since $f(z) \neq 0$ for $z \in C$, f_t'/f_t is a continuous function in both z and t, at least when z is restricted to C. One can then show that this implies n_t is continuous in t. (In other words, integrating a continuous function in two variables, where each variable is restricted to a compact set, yields a continuous function in one variable. We essentially proved this in an earlier homework assignment.) □

You should think of Rouche's Theorem as a type of perturbation statement. The hypothesis that $|f| > |g|$ means that you can think of $f + g$ as a slight perturbation of f, and the theorem states that under these small perturbations, the number of zeros of f inside C is unchanged. As an application, we can prove a quantitative version of the Fundamental Theorem of Algebra, which provides upper bounds on the size of the roots of a complex polynomial.

Theorem 4. *Suppose* $f(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ *is a complex polynomial. Then every root* z_i *of* $f(z)$ *satisfies* $|z_i| \leq 1 + \max(|a_0|, |a_1|, \ldots, |a_{n-1}|)$.

Proof. Let $g(z) = z^n$ and let $M = \max(|a_0|, \ldots, |a_{n-1}|)$. We claim that $|g(z)| \geq |f(z) - g(z)|$ on $|z| = 1 + M$.

Indeed, notice that when $|z| = 1 + M$, $|g(z)| = |z^n| = (1 + M)^n$. On the other hand, $f - g = a_{n-1}z^{n-1} + \ldots + a_0$, so

$$|f-g(z)| \leq |a_{n-1}||z|^{n-1}+\ldots+|a_0| \leq M(1+M)^{n-1}+M(1+M)^{n-2}+\ldots+M = M\frac{(1+M)^n-1}{1+M-1} = (1+M)^n$$

So $|f - g| < |g|$, as desired. Then Rouche's Theorem says $g(z), f - g + g = f(z)$ have the same number of zeros inside $|z| = 1 + M$. Since $g(z) = z^n$ has n zeros inside this circle, this means that $f(z)$ does as well, and the fundamental theorem of algebra tells us $f(z)$ has exactly n zeros in total, so all the zeros of $f(z)$ are inside this circle. □

The main idea of this proof is that sometimes it is possible to pick out the zeros of a polynomial using Rouche's theorem by identifying the dominant term on a curve of interest. In this proof, it is the highest order term, z^n, which dominates all others on $|z| = 1 + M$. We can adopt this method to sometimes count zeros of polynomials (or sometimes even more general functions) inside various curves.

Example. Let $f(z) = z^3 - 2z^2 + 5z - 1$. Count the number of zeros inside $|z| = 1$. To solve this problem using Rouche's theorem, we identify the largest term on $|z| = 1$. Notice that $|z^n| = 1$ regardless of what n is on the unit circle, so $|z^3| = 1, |-2z^2| = 2, |5z| = 5, |-1| = 1$. This means the largest term is $5z$, so let $g(z) = 5z$. Also, notice that $5z$ is large enough to dominate the sum of all other terms on $|z| = 1$, because the triangle inequality implies that $|z^3 - 2z^2 - 1| \leq |z^3| + |2z^2| + 1 = 4$ on $|z| = 1$. Therefore, if we let $h(z) = z^3 - 2z^2 - 1$, then $|g(z)| > |h(z)|$ on the unit circle, so Rouche's Theorem implies that $g(z)$ and $g + h = f$ have the same number of zeros inside $|z| = 1$. Since $5z$ has exactly one zero inside $|z| = 1$, this means $f(z)$ has exactly one zero inside $|z| = 1$. (As a matter of fact, one can use a computer to check the roots are $z \approx 0.216757, 0.891622 \pm 1.95401i$.)

Obviously this method is not foolproof, because to easily use it requires functions which have one term much larger than the rest of the function on the curve of interest, and it must be easy to compute the number of zeros of that one term inside the curve of interest.

3. Contour integration over more general curves

So far many of our general theorems about contour integrals have required some sort of hypothesis on the shape of the curve C. In particular, we often require C to be a circle, triangle, rectangle, keyhole contour, etc. What about more general shapes?

We state a theorem which provides a method for evaluating integrals over more general curves. Suppose C_0 and C_1 are two curves parameterized by $z_0(t), z_1(t), 0 \leq t \leq 1$, where $z_0(0) = a = z_1(0), z_0(1) = b = z_1(1)$. We say that C_0 and C_1 are *homotopic in* Ω if there exists a continuous function $F(t,s) : [0,1] \times [0,1] \to \Omega$ such that $F(t,0) = z_0(t), F(t,1) = z_1(t)$ and $F(0,s) = a, F(1,s) = b$ for all s. We sometimes call F a homotopy of C_0 to C_1.

You should think of $F(t,s)$ as representing a function $z_s(t)$ which parameterizes a curve in the variable t. The variable s parameterizes the deformation of z_0 to z_1. The fact that F is continuous and takes values in Ω says that the deformation of z_0 to z_1 is continuous and does not leave the set Ω. Finally, the condition that $F(0,s) = a, F(1,s) = b$ for all s corresponds to the fact that the endpoints of the curves during this deformation process remain fixed.

Examples.

- Let $\Omega = \mathbb{C}$, and let C_0 be the line segment connecting 1 to -1, and C_1 be the semicircle parameterized by $z_1(t) = e^{\pi i t}, 0 \leq t \leq 1$. Then C_0 and C_1 are homotopic in Ω. Intuitively, we can see this because we can visualize a continuous deformation of C_1 to C_0, say. Rigorously, one has $F(t,s) = (1-s)(1-2t) + se^{\pi i t}$ as a homotopy: when $s = 0$ we have $F(t,0) = (1-2t)$, which parameterizes C_0, and when $s = 1$, we have $F(t,1) = e^{\pi i t}$, which parameterizes C_1. Also, $F(t,s)$ is continuous on $[0,1] \times [0,1]$, and $F(0,s) = (1-s) + s = 1, F(1,s) = (1-s)(-1) + s(-1) = -1$.
- Suppose C_0 is the unit circle $z_0(t) = e^{2\pi i t}$, and C_1 is the stationary curve $z_1(t) = 1$. Then C_0 is homotopic to C_1 in $\mathbb{C}$. Intuitively, we can see this because we can shrink the unit circle to a point. Can you find a rigorous homotopy of C_0 to C_1?

The main theorem of interest, which we will not prove (due to shortage of time, not because the proof is advanced) is the following:

Theorem 5. *Suppose f is holomorphic in an open set Ω. Suppose C_1 is homotopic to C_2 in Ω. Then*

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz.$$

A vague sketch of the proof is as follows: we look at the image of the homotopy F in Ω. If this image were contained in an open disc contained in Ω, we could apply the theorem which tells us that f has a primitive in this open disc, and then the theorem would be easy to prove. However, in general, we are not in this situation. Nevertheless, we can try to salvage this idea by constructing small discs contained in Ω which cover the image of the homotopy, and then construct primitives on small discs. One shows that it is possible to chain together lots of small discs and use what we know about integrals of functions with primitives in small discs to prove the theorem. See the book for a detailed proof.

This key part of this theorem is that we require the homotopy stay in the domain of holomorphy of f.

Example. Let $f(z) = 1/z$. Then f is holomorphic on $\mathbb{C}-0$. Notice that $\int_{S^1} 1/z\,dz = 2\pi i$, whereas if C is the stationary curve at 1 then $\int_C 1/z\,dz = 0$. Therefore, the above theorem implies that no homotopy of S^1 to a single point 1 exists on $\mathbb{C}-0$. In other words, the fact that $\int_{S^1} 1/z\,dz \neq 0$ is reflected in the fact that the space $\mathbb{C}-0$ has a hole at 0.

Suppose Ω is an open connected set in which any closed curve γ is homotopic a stationary curve. Then we call Ω a *simply-connected (open) set*. A consequence of the above theorem is that

Corollary 2. *Suppose f is holomorphic on a simply-connected set Ω. Then*

$$\int_\gamma f(z)\,dz = 0$$

for any closed curve γ in Ω.

Another consequence of this theorem is the following:

Corollary 3. *Suppose f is holomorphic on a simply-connected set Ω. Then f has a primitive function F on Ω.*

See the textbook for a short proof; it uses similar ideas to the theorem which proves the existence of a primitive of holomorphic functions on a disc. The primitive can be defined using a contour integral; more specifically, if z_0 is an arbitrary but fixed point in Ω, then one can define

$$F(z) = \int_C f(w)\,dw,$$

where C is any curve contained in Ω which connects z_0 to z. One then shows that $F(z)$ is holomorphic on Ω and is a primitive for $f(z)$. Notice that the fact that Ω is simply-connected guarantees that $F(z)$ is the same regardless of the choice of curve C which connects z_0 to z.

These theorems raise a natural question: what types of spaces are simply-connected? In general, one might want to take a topology course to answer this question. However, we can give a few examples based on elementary knowledge right now.

Example. A *convex set* in $\mathbb{C}$ is a set Ω such that if $z_1, z_2 \in \Omega$, then the line segment joining z_1 and z_2 is still in Ω. We claim any connected convex set is simply-connected. Indeed, given a closed curve γ parameterized by $z(t)$, consider the homotopy $F(t,s) = (1-s)z(t)+sz(0)$. Then $F(t,0) = z(t)$, $F(t,1) = z(0)$, $F(0,s) = (1-s)z(0)+sz(0) = z(0)$, $F(1,s) = (1-s)z(1)+sz(1) = z(1)$. Also, F is clearly continuous, and $F(t,s) \in \Omega$, because $F(t,s)$ is a point on the line segment joining $z(t)$ to $z(0)$.

So, for example, $\mathbb{C}$ is clearly convex, hence is simply-connected. Other convex sets include rectangles, triangles, circles, and ellipses. Notice, however, that keyhole contours and arbitrary polygons (even quadrilaterals) are not necessarily convex. Despite this fact, one can still show, using rigorous techniques from topology (in particular, an idea known as homotopy-equivalence), that keyhole contours and interiors of polygons are simply-connected.

4. Defining the logarithm

We can use what we just learned about simply-connected sets and integrals to define the logarithm on certain subsets of $\mathbb{C}$. We would like the following to be true of a log function: if $f(z) = \log z$, then $f'(z) = 1/z$, and we would also like $e^{\log z} = z$. Notice that it is probably unreasonable to ask $\log e^z = z$, at least for all z, because e^z is not an injective function: that is, there are many values of z for which e^z is the same. More specifically, $e^{z_1} = e^{z_2}$ if and only if $z_1 - z_2 = 2n\pi i$, for some integer n. If it is possible to find a f for which all of the above are true on a subset Ω of $\mathbb{C}$, then we say that it is possible to define a log function on Ω.

Because e^z is not injective, we immediately see that if it is possible to define a log function, there will be multiple possible choices for the value of $\log z$. For example, $\log 1$ could be equal to any value $2n\pi i$, since $e^{2n\pi i} = 1$. If it is possible to define log on a set Ω, we will want to choose the values of log in such a way to ensure that log is continuous on Ω.

If we want the derivative of $\log z$ to be $1/z$, then we already know that it will be impossible to define log on any subset of $\mathbb{C} - 0$ which contains a circle surrounding the origin. Therefore, we will only be able to define log functions on certain subsets Ω of $\mathbb{C} - 0$.

Suppose Ω is a simply-connected open set which does not contain 0 but contains 1. Then $1/z$ is holomorphic on all of Ω, then by the previous section we can define a holomorphic function

$$F(z) = \int_C \frac{1}{w}\, dw,$$

where C is any curve connecting 1 to z in Ω. Then $F'(z) = 1/z$ is immediate, and also $F(1) = 0$. We now want to show that $e^{F(z)} = z$. To do so, consider the derivative of $ze^{-F(z)}$. Since $F(z)$ is holomorphic on Ω, so is $ze^{-F(z)}$, and we have

$$\frac{d}{dz} ze^{-F(z)} = e^{-F(z)} - F'(z)ze^{-F(z)} = 0.$$

Therefore, by a theorem we proved much earlier, $ze^{-F(z)}$ is constant on Ω. Suppose $ze^{-F(z)} = c$. First, notice that $c = 0$ is impossible, because if $ze^{-F(z)} = 0$ on all of Ω, then $e^{-F(z)} = 0$, which is impossible since $e^{-F(z)}$ is never 0. Therefore $e^{F(z)} = z/c$. If we plug in $z = 1$, we find that $e^{F(1)} = 1/c$, or $1 = 1/c$, or $c = 1$. Therefore $e^{F(z)} = 1$.

In summary, we have proven the following:

Theorem 6. *If Ω is an open simply-connected set containing 1 but not 0, and we define*

$$F(z) = \int_C \frac{1}{w}\, dw$$

where C is any curve connecting 1 to z contained in Ω, then $F(z)$ is holomorphic on Ω, $F'(z) = 1/z$, and $e^{F(z)} = 1$.

Example. Let Ω be $\mathbb{C}$ with the negative real axis removed (ie, $\mathbb{C}$ with points $-x$ removed, where $x \geq 0$.) Then Ω is simply-connected (see exercise 21b in the text about star-shaped domains for a suggestion on how to prove this fact), contains 1, and does not contain 0. The above function $F(z)$ for this choice of Ω is sometimes called a *principal branch* of the logarithm function. This terminology refers to the fact that, a priori, there are many possible choices for log on this domain, because of the fact that e^z is periodic with period $2\pi i$. We select one of these choices.

This choice for log is called the principal branch because $F(z)$ agrees with the usual $\log z$ when $z > 0$ is a positive real number. Indeed, given arbitrary $z > 0$ positive real, we can select C to be the straight line segment connecting 1 to z, and then $F(z) = \int_1^z 1/w \, dw$, which is how the log function is usually defined in real calculus.

More generally, for this choice of the branch of log, given $z = re^{it}$, where t is chosen to satisfy $-\pi < t < \pi$, one has $\log z = \log r + it$. Indeed, we can choose C to be the piecewise-smooth curve consisting of the straight line segment from 1 to r, and then the circular arc from r to re^{it}. The integral over the first part of this curve gives $\log r$, and one easily checks the integral over the circular arc gives it.

For example, assuming that log denotes the principal branch of the logarithm function, we have $\log 1 + i = \log \sqrt{2}e^{i\pi/4} = \log \sqrt{2} + i\pi/4$. Similarly, $\log i = \log 1 \cdot e^{\pi i/2} = \log 1 + i\pi/2 = i\pi/2$.

More generally, it is not necessary that Ω be a simply-connected set which contains 1. We can let Ω be any simply-connected set which does not contain 0, and then alter the definition of $F(z)$ accordingly to ensure that $e^{F(z)} = z$. In particular, when we define C as any curve connecting 1 to z, we replace 1 with an arbitrary point z_0 in Ω, and then we define $F(z) = \int_C f(w) \, dw + c$, where c is a constant such that $e^c = z_0$. (Indeed, if $z = z_0$, then $F(z_0) = 0 + c$, so we want $e^{F(z_0)} = e^c = z_0$.)

The only possible ambiguity in the definition of a log function is due to the fact that e^z is periodic with period $2\pi i$. More specifically, if Ω is an open set (simply-connected or not) in which it is possible to define $F(z), G(z)$ which are both log functions (in the sense that $F'(z) = G'(z) = 1/z$ and $e^{F(z)} = z, e^{G(z)} = z$ for all $z \in \Omega$, then $F(z) - G(z) = 2n\pi i$ for some integer n, for all $z \in \Omega$. Indeed, the fact that $e^{F(z)} = e^{G(z)} = z$ on all of Ω implies that $F(z) = G(z) + 2\pi i n(z)$ for some function $n(z)$ defined on Ω which only takes integer values. Since F, G are both holomorphic, hence continuous, it must be the case that $n(z)$ is constant on Ω, hence equal to some integer n on all of Ω.

It is important to remember that when speaking of the logarithm function for complex numbers, you must always keep in mind two facts: first, the logarithm cannot be defined on all subsets of $\mathbb{C}$, but it can be defined for simply-connected subsets containing 1 but not 0 (or any subsets of such subsets), and that this definition always comes with a choice of branch of the log function. This choice must be specified for the log function to be unambiguous; if Ω is the slit plane above, then the principal branch is frequently chosen (since it agrees with the usual log function on positive reals). But in other situations the choice might not be so obvious; for example, if Ω is a small disc centered at a negative real number. In this case when speaking of log be sure to make completely explicit what branch of the log you are choosing.

Example. Suppose Ω is the slit plane, and we define $\log z$ not to be the principal branch, but the principal branch plus $4\pi i$. Then $\log 1 = 4\pi i$, and $\log i = 4\pi i + i\pi/2 = 9\pi i/2$. Notice that $e^{\log 1} = e^{4\pi i} = 1$ is still satisfied, as is $e^{\log i} = e^{9\pi i/2} = i$.

Finally, as a nice application of the logarithm function, we can define arbitrary powers of complex numbers on any subset Ω on which it is possible to define a log function. Let α be an arbitrary complex number. Fix a branch of the logarithm on Ω. Then we can let $z^\alpha = e^{\alpha \log z}$. In many situations, $1 \in \Omega$, and we want to choose the principal branch of the logarithm, although this is not always necessary.

Example. Let $\alpha = 1/2$, and Ω be the slit plane. Choose the principal branch of log on Ω. If $z = re^{it}$ with $-\pi < t < \pi$, then $z^{1/2} = e^{1/2 \log z} = e^{1/2 \log r + it} = e^{1/2 \log r} \cdot e^{it/2} = \sqrt{r} \cdot e^{it/2}$. Notice that $(\sqrt{r} \cdot e^{it/2})^2 = re^{it} = z$, as expected. For example, with this choice of branch of log, we have $i^{1/2} = e^{1/2 \log i} = e^{1/2 \cdot \pi i/2} = e^{\pi i/4} = 1/\sqrt{2} + 1/\sqrt{2}i$. Similarly, $-i^{1/2} = 1/\sqrt{2} - 1/\sqrt{2}i$. Interestingly enough, notice that for this choice of Ω, $-1^{1/2}$ is undefined.

If we had chosen the branch of the log obtained by adding $2\pi i$ to the principal branch, we would have obtained a different function $z^{1/2}$. For example, $1^{1/2} = e^{1/2 \log 1}$. If $\log 1 = 0$ (principal branch), then $1^{1/2} = e^0 = 1$, as expected. However, if we add $2\pi i$ to the principal branch of log, then $1^{1/2} = e^{1/2 \log 1} = e^{1/2 \cdot 2\pi i} = e^{\pi i} = -1$. Therefore we see that choosing different branches of logs corresponds to the fact that frequently fractional powers are not uniquely defined.

Furthermore, notice that it is impossible to define $z^{1/2}$ as a holomorphic function on all of $\mathbb{C}$. A rigorous proof of this is left as an exercise.