THE ARITHMETIC OF COMPLEX NUMBERS

Contents

Before doing calculus on complex functions, we need to have a good understanding of the basic properties of complex numbers. In particular, we need to know how to perform arithmetic on complex numbers!

1. Basic properties of complex numbers

A *complex number* is a number of the form $x + yi$, where $x, y \in \mathbb{R}$ and $i^2 = -1$. Let $z = x + yi$. Then we often call x the *real part* of z, and sometimes write $\operatorname{Re} z$ for x, and we call y the *imaginary part* of z, and sometimes write $\operatorname{Im} z$ for y. (Even though y is the imaginary part of z, y is a real number!)

We often sketch complex numbers by representing the number $z = x + iy$ as the point (x, y) in the *complex plane*, which graphically looks just like $\mathbb{R}^2$. The x-axis is often referred to as the *real axis*, and the y-axis is referred to as the *imaginary axis*.

The two basic arithmetic operations on complex numbers are addition and multiplication (and their relatives, subtraction and division). If $z_1 = x_1 + y_1 i, z_2 = x_2 + y_2 i$ are two complex numbers, then their sum is just the complex number $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$ obtained by summing the real parts together and the complex parts together, respectively. To multiply z_1, z_2, we distribute appropriately and use the fact that $i^2 = -1$:

$$z_1 z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i. \tag{1}$$

We can also negate a complex number by negating its real and imaginary parts, respectively; namely, $-z = -x + (-y)i$. We can then define $z_1 - z_2$ as $z_1 + (-z_2)$. Obviously $z + (-z) = 0$, where 0 is thought of as $0 + 0i$. Taking the reciprocal of a complex number is slightly more complicated. One can check that if $z = x + yi$ and $z \neq 0$, then

$$\frac{1}{z} = \frac{1}{x + yi} = \frac{x - yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i. \tag{2}$$

Indeed, in the above calculation we eliminate i from the denominator by multiplying $x + yi$ by $x - yi$, which gives $x^2 + y^2$. One easily checks that $z \cdot \frac{1}{z} = 1$.

As a matter of fact, the number $x^2 + y^2$ is quite special; we call $|z| = \sqrt{x^2 + y^2}$ the *absolute value* or *modulus* of z. Just as how $|x|$ can be thought of as a measure of the size of x when x is a real number, we can think of $|z|$ as a measure of the size of z. More generally, if z_1, z_2 are complex numbers, then $|z_1 - z_2|$ can be thought of as the distance from z_1 to z_2.

How are each of these operations reflected geometrically in the complex plane? Addition of complex numbers $x_1 + y_1 i, x_2 + y_2 i$ is evidently the same as vector addition of the two vectors $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle$. Similarly, the negative of z is graphically represented by reflecting z through the origin. The absolute value of z is clearly the distance of the point representing z from the origin.

To get a good intuitive description of multiplication requires that we use an alternate description of complex numbers. One of the most remarkable facts about complex numbers is the identity $e^{it} = \cos t + i \sin t$, where $t \in \mathbb{R}$. We can basically see why this is true by thinking about the power series representations of $e^t, \cos t, \sin t$:

$$\begin{aligned} e^t &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots, \\ \cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \dots, \\ \sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots. \end{aligned}$$

Replacing t with it in e^t, using the fact that $i^2 = -1$, and then collecting the real and imaginary parts, respectively, give the identity $e^{it} = \cos t + i \sin t$. It is worth pointing out that this is not entirely rigorous, because we have not yet proven that power series make sense when we allow the variable to be a complex instead of real. We will make this rigorous over the next few classes.

Since $\cos^2 t + \sin^2 t = 1$, we see that $|e^{it}| = 1$ for all $t \in \mathbb{R}$. That is, the numbers e^{it} are represented exactly by the points on the unit circle in the complex plane. So given any $z \neq 0$, the number $z/|z|$ is a number of absolute value 1, hence lies on the unit circle, so we can write $z/|z| = e^{i\theta}$ for some $\theta \in \mathbb{R}$. Therefore any nonzero z can be represented as $z = |z|e^{i\theta} = re^{i\theta}$, where we let $r = |z|$. Notice that r is uniquely determined by z, and θ is uniquely determined up to an integer multiple of 2π. We call a representation of a complex number $z = x+yi$ in the form $re^{i\theta}$ the *polar representation* of z. Indeed, one can just think of (r, θ) as the polar coordinates of the point (x, y). The number θ is often called the *argument* of z (we are abusing grammar here since strictly speaking the argument is only determined up to an integer multiple of 2π), and we sometimes write arg z for the argument of z.

Notice that if $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$, then $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$. Given a positive real number c, it is clear that multiplying z by c corresponds to scaling the point z by a factor of c. Similarly, given a complex number e^{it} of unit modulus, multiplying $z = re^{i\theta}$ by e^{it} gives $re^{i(\theta+t)}$, hence corresponds to a rotation of z about the origin through t radians, in a counterclockwise direction. Therefore, we can geometrically think of multiplication of numbers by multiplying the absolute values together, and then adding the arguments together. Multiplying the absolute values corresponds to a scaling, while adding the arguments corresponds to a rotation.

Examples.

- If $z = 1$, then the polar representation of 1 is $1 = 1e^{i\cdot 0}$. Similarly, if $z = i$, then $i = 1e^{i\cdot\pi/2}$.
- If $z = 2 + 2i$, then $2 + 2i = 2\sqrt{2}e^{i\pi/4}$.
- Let's solve for z in the equation $e^z = 1$. Obviously $z = 0$ is a solution, but it is not the only complex solution. If we write $z = x + yi$, then $e^z = e^x e^{iy} = e^x(cosy + i\sin y)$. For this to equal 1, we need $e^x = 1$, and we also need $e^{iy} = 1$. Clearly $e^x = 1$ only has solution $x = 0$. On the other hand, $e^{iy} = 1$ has infinitely many solutions; namely, all $y = 2\pi in$, where $n \in \mathbb{Z}$. Therefore the solutions of $e^z = 1$ are $z = 2\pi in$, where $n \in \mathbb{Z}$.
- Any complex number ζ_n which satisfies $z^n = 1$ is called a *complex nth root of unity*. If $\zeta_n = re^{i\theta}$, then we must have $r^n e^{i\theta n} = 1$. However, this is only possible if $r = 1$ (we can conclude $r = 1$ from $r^n = 1$ since we know that $r > 0$ is a positive real number) and if θn is an integer multiple of 2π. Therefore, the nth roots of unity are the numbers

$$1 = e^{2\pi i\cdot 0/n}, e^{2\pi i/n}, e^{2\pi i\cdot 2/n}, \ldots, e^{2\pi i\cdot(n-1)/n}; \tag{3}$$

  that is, numbers of the form $e^{2\pi ik/n}$, where k is any positive integer. There are n such numbers, and if we plot them in the complex plane, we see that they are the vertices of a regular n-gon inscribed in the unit circle.
- A clever application of the identity $e^{it} = \cos t + i\sin t$ is in helping to memorize trigonometric identities; in particular, identities for $\cos nt, \sin nt$, where n is a positive integer. For example, suppose we want to remember the double-angle identities for $\cos 2t, \sin 2t$. Then

$$(e^{it})^2 = e^{2it} = \cos 2t + i\sin 2t,$$

  but $(e^{it})^2$ is also equal to

$$(\cos t + i\sin t)^2 = (\cos^2 t - \sin^2 t) + i(2\sin t\cos t).$$

  Since these two complex numbers are equal, the real and imaginary parts of these two numbers must be equal, so $\cos 2t = \cos^2 t - \sin^2 t, \sin 2t = 2\sin t\cos t$. The idea is the same for $\cos nt, \sin nt$, though to be obtain identities for these functions, we will need to expand $(cost + i\sin t)^n$, which in general requires the binomial theorem.

Suppose $z = x + yi$ is a complex number. Then the *complex conjugate* of z is the number $\overline{z} = x - yi$. Geometrically, complex conjugation corresponds to reflection across the x-axis. If $z = re^{i\theta}$, then $\overline{z} = re^{i(-\theta)}$. Notice that $z\overline{z} = (x + yi)(x - yi) = x^2 + y^2 = |z|^2$. Notice that if z is real, then $\overline{z} = z$. A very useful property of complex conjugation is that it commutes with addition and multiplication: that is,

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \overline{z_1 z_2} = \overline{z_1}\cdot\overline{z_2}. \tag{4}$$

One application of this property is showing that the complex roots of any real polynomial always come in complex conjugate pairs.

Example. If $f(x) = a_0 + a_1 x + \ldots + a_n x^n$ is a real polynomial (that is, all the a_i are real), and if z is a root of $f(x)$, then $\overline{z}$ is also a root of $f(x)$. Indeed, if $f(z) = 0$, then

$$a_0 + a_1 z + \ldots + a_n z^n = 0.$$

If we apply complex conjugation to both sides and apply the fact that conjugation commutes with addition and multiplication repeatedly, together with the fact that all the a_i are real, we get

$$\overline{a_0} + \overline{a_1} \cdot \overline{z} + \overline{a_2} \cdot \overline{z}^2 + \ldots + \overline{a_n} \cdot \overline{z}^n = 0.$$

Since $\overline{a_i} = a_i$, this shows that $\overline{z}$ is also a root of $f(x)$, as desired.

Finally, we mention a frequently useful inequality called the *triangle inequality*, which states that if z_1, z_2 are any complex numbers, then

$$|z_1| + |z_2| \geq |z_1 + z_2|, \tag{5}$$

with equality if and only if z_1, z_2 are non-negative real multiples of each other. This is a consequence of the usual triangle inequality, since we can interpret $0, z_1, z_1 + z_2$ as the vertices of a triangle with side lengths $|z_1|, |z_2|, |z_1 + z_2|$.

2. An index to properties, facts, etc.

- Multiplying complex numbers (Equation 1)
- Reciprocal of a (nonzero) complex number (Equation 2)
- Roots of unity (Equation 3)
- Properties of complex conjugation (Equation 4)
- Triangle inequality (Equation 5)