

# TOPOLOGY, LIMITS OF COMPLEX NUMBERS

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### 1. TOPOLOGY AND LIMITS OF COMPLEX NUMBERS

Since we will be doing calculus on complex numbers, not only do we need to know how to do arithmetic on complex numbers, we also need to understand how to take limits in complex numbers. Fortunately, the situation is very similar to what happens in  $\mathbb{R}$ .

For example, suppose  $z_1, z_2, \dots, z_n, \dots$  is an infinite sequence of complex numbers. We say that this sequence *converges* to the complex number  $w$  if  $\lim_{n \rightarrow \infty} |z_n - w| = 0$ , and write  $\lim_{n \rightarrow \infty} z_n = w$  if this is the case. Notice that in this definition, the numbers  $|z_n - w|$  are all real, so this definition is not circular.

One can easily check that  $z_n \rightarrow w$  if and only if  $\operatorname{Re} z_n \rightarrow \operatorname{Re} w, \operatorname{Im} z_n \rightarrow \operatorname{Im} w$ . Therefore, evaluating limits of sequences of complex numbers is, in practice, the same thing as evaluating the limits of two sequences of real numbers. Geometrically, a sequence of numbers  $z_n$  in the complex plane approaches a limit  $w$  if and only if the points  $z_n$  get closer and closer to  $w$ .

**Example.** Suppose  $z_n = \frac{n}{n+1} + \frac{\sin n}{\sqrt{n}}i$ . Then  $\operatorname{Re} z_n = \frac{n}{n+1}$  has limit 1, and  $\frac{\sin n}{\sqrt{n}}$  has limit 0, so  $z_n \rightarrow 1$ .

As a matter of fact, since  $|z_1 - z_2|$  is just equal to the distance of  $z_1, z_2$  from each other in the complex plane, taking absolute values of complex numbers gives the same distance function as ordinary Euclidean distance in  $\mathbb{R}^2$ . If you have heard of a *metric space* before, the function  $d(z_1, z_2) = |z_1 - z_2|$  defines a *metric* on  $\mathbb{C}$ . If you haven't heard of metrics before, intuitively speaking, a metric provides a way to measure distance on a set. Metrics must satisfy certain axioms, which we omit here, and one can prove a variety of fairly general facts about metric spaces. This is often covered in a topology or real analysis class.

Since the absolute value on  $\mathbb{C}$  defines a metric on  $\mathbb{C}$ , we can talk about various topological notions on  $\mathbb{C}$ , like open sets, closed sets, compactness, connectedness, etc. Again, all of these notions are defined in the more general setting of a topological space, which is covered in a first class on topology, but we do not assume prior knowledge of that subject here. Instead we will just give the specific definitions and properties (omitting proofs if they are long) that we need.

Suppose  $z_0$  is a fixed complex number and  $r$  an integer. Then the *open disc* or *open ball* centered at  $z_0$  of radius  $r$  is the set of complex numbers

$$D_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

Geometrically, an open ball centered at  $z_0$  is just an open disc centered at  $z_0$  in the complex plane. Notice that the inequality  $|z - z_0| < r$  is strict; this is in contrast to a *closed ball* centered at  $z_0$  of radius  $r$ , which is the set of complex numbers

$$\overline{D}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\}.$$

This is just a closed disc of radius  $r$  in the complex plane centered at  $z_0$ . Notice that the closed disc contains all of its boundary points, while the open disc contains none of its boundary points. We will repeatedly use open and closed discs for the remainder of this class, so that is why we use shorthand notation for these sets.

Let  $\Omega$  be a subset of  $\mathbb{C}$ . Given any  $z_0 \in \Omega$ , we call  $z_0$  an *interior point* of  $\Omega$  if there exists some  $r_0 > 0$  such that  $D_{r_0}(z_0) \subset \Omega$ . Intuitively,  $z_0$  is an interior point of  $\Omega$  if it belongs to  $\Omega$  but does not lie on its boundary. For example, one quickly checks that any point on the boundary of a closed disc is not an interior point, because any open ball centered at such a point will always contain points outside the closed disc.

If  $\Omega$  is a subset of  $\mathbb{C}$  such that every point of  $\Omega$  is an interior point, then we call  $\Omega$  an *open set*. If  $\Omega$  is a subset of  $\mathbb{C}$  such that  $\mathbb{C} - \Omega$  (the complement of  $\Omega$  in  $\mathbb{C}$ ) is open, then we call  $\Omega$  an *closed set*. Intuitively speaking, open sets contain none of the points on their boundary, while closed sets contain all of the points on their boundary. Both the empty set  $\emptyset$  and all of  $\mathbb{C}$  are considered both open and closed.

### Examples.

- Open discs are open sets. Indeed, given any point  $z_0$  in an open disc  $\Omega$  of radius  $r$ , we can find some  $r_0$  small such that  $D_{r_0}(z_0) \subset \Omega$ . (For example, choose  $r_0$  to be half the difference of the distance of  $z_0$  from the center of  $\Omega$  and  $r$ .)
- Sets like  $\operatorname{Re} z > c, \operatorname{Im} z > c$  where  $c$  is a constant real number are open sets.
- Closed discs are closed sets. Indeed, it is obvious that the complement of a closed disc is open.
- Sets like  $\operatorname{Re} z \geq c, \operatorname{Im} z \geq c$  are closed sets.
- Finite collections of single points are closed. Obviously, there are some infinite collections of single points which are not closed.
- There are many sets which are neither open nor closed. For example, the open disc of radius 1 centered at 0 together with 1 is neither open nor closed.

There is an alternate characterization of closed sets which is frequently useful. Let  $\Omega$  be any subset of  $\mathbb{C}$ . If  $z_n$  is a sequence of points in  $\Omega$  converging to some point  $z \in \mathbb{C}$  with  $z_n \neq z$  for all  $n$ , then we call  $z$  a *limit point* of  $\Omega$ . Intuitively, a limit point of a set is a point which can be approximated arbitrarily closely by other points in that set. Notice that a limit point of  $\Omega$  may or may not be a point of  $\Omega$  itself.

Then one can show that a closed set  $\Omega$  is a set which contains all of its limit points (indeed, this is done in introductory topology classes). In other words, any point which can be approximated arbitrarily closely by elements of a closed set must belong to the closed set itself. More generally, if  $\Omega$  is any set, then the union of  $\Omega$  with all of its limit points is called the *closure* of  $\Omega$ , and is written  $\overline{\Omega}$ . Again, many properties of the closure of a set are proven in an introductory topology class. For

example, one can show that the closure of any set  $\Omega$  is closed (hence the name), and that it is also the smallest closed set containing  $\Omega$ , in the sense that it is the intersection of all closed sets containing  $\Omega$ . The *boundary* of a set  $\Omega$  is the closure of  $\Omega$  minus its interior, and is often written  $\partial\Omega$ .

**Example.** Consider the open disc  $\mathbb{D} = D_1(0)$  of radius 1 centered at the origin. This set is evidently not closed. Notice that the sequence of points  $0, 1/2, 2/3, 3/4, \dots$  in  $\mathbb{D}$  has limit point 1, which is not in  $\mathbb{D}$ . The closure of  $\mathbb{D}$  is the closed disc of radius 1 centered at the origin.

Let  $\Omega$  be any set in  $\mathbb{C}$ . We say that  $\Omega$  is *bounded* if there exists some real number  $M$  such that  $|z| < M$  for all  $z \in \Omega$ . In other words, no part of  $\Omega$  gets infinitely far from the origin. Alternately,  $\Omega$  is bounded if we can find a (large) disc which contains all of  $\Omega$ . Also, if  $\Omega$  is bounded, then we define the *diameter* to be the supremum of  $|z_1 - z_2|$  as  $z_1, z_2 \in \Omega$ . In other words, the diameter of  $\Omega$  is the smallest real number which upper bounds the distances of any two points in  $\Omega$ .

### Examples.

- A closed disc of radius  $r$  has diameter  $2r$ , since any two points in such a disc are separated by a distance at most  $2r$ , and the distance of  $2r$  is actually achieved by points on opposite ends of a diameter of the disc.
- The diameter of a set does not actually have to be achieved as the distance between two points in that set. For example, an open disc of radius  $r$  also has diameter  $2r$ , but there are no two points in such a disc whose distance is equal to  $2r$ . Nevertheless, we can find points in the disc whose distances get arbitrarily close to  $2r$ .

If  $\Omega$  is both closed and bounded, then we call  $\Omega$  *compact*. Compact sets have many special and useful properties, which are again proven in introductory topology classes. We list them here without proof.

- Let  $U_\alpha$  be a collection of open sets whose union contains a compact set  $\Omega$ . Then there exists a finite subcollection of the  $U_\alpha$  whose union contains  $\Omega$ . (This is often phrased as saying that any open cover of a compact set contains a finite subcover. This property is usually taken as the definition of a compact set.)
- If  $\Omega$  is a compact set in  $\mathbb{C}$ , then any sequence of points in  $\Omega$  contains a subsequence which converges to some point in  $\Omega$ . (This is Theorem 1.2 of the text.)
- A property which we will make use of in a few weeks is the following: suppose we have a nested sequence of non-empty compact sets  $K_1 \supset K_2 \supset K_3 \dots$  with  $\text{diam}(K_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then there exists a unique point  $w$  contained in every  $K_n$ . For a proof, see the text (Proposition 1.4).

### Examples.

- We will show that each of the three properties fails for suitable choices of non-compact sets. For example, let  $\Omega$  be the open disc of radius 1 centered at 0. Then consider the open cover of  $\Omega$  consisting of the discs  $D_{(n-1)/n}(0)$  of

radius  $1 - 1/n$  centered at 0. Every point of  $\Omega$  lies in one of these discs, so this does indeed form an open cover, but there is no finite subcover, because any finite subcover will contain a disc of largest radius  $1 - 1/k$ , say, and then any point of  $\Omega$  of distance between  $1 - 1/k$  and 1 will not lie in this finite subcover. This argument fails for the closed disc of radius 1 since then the collection of open sets  $D_{(n-1)/n}(0)$  does not contain any of the points on the boundary of  $\Omega$ .

- It is easy to think of a sequence of points in the open disc  $\Omega$  above which has no subsequence converging to a point of  $\Omega$ . For example,  $0, 1/2, 2/3, 3/4, \dots$  works, as the only possible limit point of this sequence is 1, which does not lie in  $\Omega$ .
- Finally, it is possible to have a nested sequence of open sets with diameter approaching 0 whose intersection contains no points. For example, let  $U_n = (0, 1/n) \times (0, 1/n)$ . These are nested open sets whose diameter approaches 0, but there is no point inside each of these sets.

There is one final important topological definition which we will frequently use. Let  $\Omega$  be an open set. We call  $\Omega$  *connected* if it is not possible to find two disjoint, non-empty open sets  $\Omega_1, \Omega_2 \subset \Omega$  such that  $\Omega = \Omega_1 \cup \Omega_2$ . Intuitively speaking, a connected set contains a single ‘piece’ (the proper term is ‘connected component’). Alternately, we say that an open set  $\Omega$  in  $\mathbb{C}$  is *path-connected* if given any two points  $z_1, z_2 \in \Omega$ , we can find a *path* connecting  $z_1$  to  $z_2$  contained entirely in  $\Omega$ . (Formally speaking, a path is a continuous function  $\gamma : [0, 1] \rightarrow \mathbb{C}$ , and its graph is the path connecting  $z_1$  to  $z_2$ .) For open sets in  $\mathbb{C}$ , the properties of being connected and path-connected are equivalent. As a matter of fact, the property of an open set  $\Omega$  being connected is equivalent to the fact that any two points in  $\Omega$  can be joined by a path consisting only of horizontal and vertical line segments.

It is possible to define a notion of connectedness for arbitrary sets, but for our purposes we will only need to know what connected means for closed sets. A closed set  $\Omega \subset \mathbb{C}$  is connected if it is impossible to find two non-empty disjoint closed sets  $\Omega_1, \Omega_2 \subset \mathbb{C}$  such that  $\Omega = \Omega_1 \cup \Omega_2$ .

### Examples.

- Open and closed discs are connected. The half-planes  $\operatorname{Re} z > c, \operatorname{Im} z > c$  are connected. However, the set consisting of points  $z$  such that  $\operatorname{Re} z > 1$  or  $\operatorname{Re} z < 0$  is not connected.
- The annulus  $1 < |z| < 2$  is connected.
- The interiors of polygons are connected.

We will very frequently find ourselves studying functions defined on an open connected non-empty set  $\Omega$ . For the rest of this class, we will call an open connected set in  $\mathbb{C}$  a *region* of  $\mathbb{C}$ . Be aware that outside of this class, it is entirely possible that a region can refer to a set which might not be open (or, more rarely, not connected)! But for the entirety of this class (and in the textbook), a region will always be an open connected non-empty set in  $\mathbb{C}$ .

This concludes the topology which we will need. Even though there is a lot we are skimming over, an intuitive grasp of the definitions and familiarity with the examples

listed above (try to think of a few other examples which illustrate these definitions on your own) should be sufficient for this class.