

# POWER SERIES

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We will now discuss basic properties of power series, which should be familiar from single-variable calculus. The difference is that we will now think of power series as defining complex functions. Power series will provide a large source of holomorphic functions, and we will see that power series play a key role in understanding properties of holomorphic functions.

## 1. INTRODUCTION TO POWER SERIES

In real calculus, a *power series* centered at a real number  $a$  is an expression of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n,$$

where the  $a_n$  are real numbers. A power series can be thought of as a generalization of a polynomial, but unlike polynomials power series do not necessarily converge at all points  $x$ . Indeed, a large part of single-variable calculus is devoted to answering questions of when power series converge.

In complex analysis, we define power series in a formally identical way to the real case: namely, a power series centered at a complex number  $z_0$  is an expression of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n,$$

where now we permit  $a_n$  to be complex numbers. Notice that for any complex number  $z$ , this infinite series is a series of complex numbers, which either converges or diverges. A natural question is to determine the set of complex numbers  $z$  for which a given power series converges. For convenience, we will let  $z_0 = 0$  for the rest of this discussion on power series, although everything works for power series centered at an arbitrary  $z_0$ .

### Examples.

- Any complex polynomial  $f(z) = a_0 + \dots + a_n z^n$  can be thought of as a power series which converges for all  $z$ .

- The power series  $1 + z + z^2 + \dots$  converges on the open disc  $|z| < 1$ , and is equal to  $1/(1 - z)$  on this disc. Indeed, since

$$1 + z + \dots + z^n = \frac{1 - z^{n+1}}{1 - z},$$

this series only converges if  $z^{n+1}$  converges to a limit as  $n \rightarrow \infty$ , and this only happens if  $|z| < 1$ , or if  $z = 1$ . However if  $z = 1$  then it is clear that  $1 + z + z^2 + \dots$  does not converge.

- The power series

$$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$$

is equal to the exponential function on the real line. As a matter of fact, we can define  $e^z$  for complex  $z$  by declaring it equal to this power series. We will return to this idea later.

Recall from real calculus that every power series has a ‘radius of convergence’  $R$ , which is a real number (or possibly infinity) such that  $\sum a_n x^n$  converges absolutely if  $|x| < R$  and diverges if  $|x| > R$ . A generalization of this fact holds true for complex power series:

**Theorem 1.** *Let  $f(z) = \sum a_n z^n$  be any power series. Then there exists an  $R$  which is either a non-negative real number or infinity, such that  $f$  converges absolutely if  $|z| < R$  and  $f$  diverges if  $|z| > R$ . Furthermore,  $1/R$  is equal to  $\limsup |a_n|^{1/n}$ . (If  $R = 0$ , we interpret the limsup as tending towards infinity, and if  $R = \infty$ , we interpret this limsup as equalling 0.) We call the open disc  $|z| < R$  the disc of convergence of  $\sum a_n z^n$ .*

*Proof.* Let  $L = 1/R$ . Suppose  $|z| < R$ . We want to show that  $\sum |a_n| |z|^n$  converges. Since  $|z| < R$ , we can choose a  $\varepsilon > 0$  such that  $(L + \varepsilon)|z| < 1$ . Since  $L = \limsup |a_n|^{1/n}$ , this means that for sufficiently large  $n$ ,  $|a_n|^{1/n} < (L + \varepsilon)$ , or  $|a_n| < (L + \varepsilon)^n$ . Therefore,

$$\sum |a_n| |z|^n \leq \sum (L + \varepsilon)^n |z|^n \leq \sum ((L + \varepsilon)|z|)^n.$$

However, the last series is a convergent geometric series. Therefore  $\sum |a_n| |z|^n$  converges.

Now suppose  $|z| > R$ . By the definition of limsup, for any  $\varepsilon > 0$  there exists infinitely many  $a_n$  satisfying  $|a_n|^{1/n} > (L - \varepsilon)$ , or  $|a_n| > (L - \varepsilon)^n$ . Choose  $\varepsilon > 0$  small enough such that  $(L - \varepsilon)|z| > 1$ . Then  $|a_n| |z|^n > (L - \varepsilon)^n |z|^n$  for infinitely many  $n$ , and because  $(L - \varepsilon)|z| > 1$ , this implies that  $a_n z^n$  does not converge to 0 as  $n \rightarrow \infty$ . Therefore  $\sum a_n z^n$  must diverge.  $\square$

Notice that the basic principle in this proof is to try to bound a series which we know very little about by a geometric series, which we fully understand. This is a frequent technique in analysis.

Exercise 17 in the textbook shows that if  $\lim |a_{n+1}|/|a_n|$  approaches a limit  $L$ , then  $\limsup |a_n|^{1/n} = L$  as well, so if the ratio test is usable (ie, if the limit of ratios of terms exists), then we can use it to find the radius of convergence. This is usually what happens in calculus classes (although sometimes the root test is useful as well).

**Examples.**

- Consider the power series  $1 - z^2 + z^4 - \dots = \frac{1}{1+z^2}$ . Since  $|a_n| = 1$  if  $n$  is even and 0 if  $n$  is odd,  $\limsup |a_n|^{1/n} = 1$ , so this power series converges for all  $|z| < 1$ . Notice that viewed as a geometric series of a complex variable, that this power series has radius of convergence 1 makes sense: when it converges it equals the function  $1/(1+z^2)$ , which is undefined at (really, getting arbitrarily large near)  $z = \pm i$ , so it would be impossible for this power series to have radius of convergence larger than 1 since it would not even be defined at the points  $\pm i$ .
- Consider the power series for  $e^z = 1 + z + z^2/2! + z^3/3! + \dots$ . Since  $a_n = 1/n!$ , and  $\lim |a_{n+1}|/|a_n| = 1/(n+1) = 0$ , this means that this power series has radius of convergence  $R = \infty$ , so this power series converges for all  $z$ . Similarly, the power series for  $\cos$  and  $\sin$  converge for all  $z$ .
- Alternately, in the power series for  $e^z$ , if one wants to compute  $|a_n|^{1/n}$  and directly use the root test, one needs to be able to evaluate the asymptotic behavior of  $(n!)^{1/n}$ . This is done by means of *Stirling's formula*, which states that

$$\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi n} n^{n+1/2} e^{-n}} = 1.$$

(In general, if  $\lim f(n)/g(n) = 1$ , we frequently write  $f(n) \sim g(n)$ .) Therefore

$$\lim_{n \rightarrow \infty} (n!)^{1/n} = \lim_{n \rightarrow \infty} \sqrt{2\pi}^{1/n} e^{-1} n^{1+1/(2n)} = \infty,$$

which shows that the power series for  $e^z$  is indeed convergent for all  $z$ .

- Determining whether a power series converges for  $|z| = R$  is a delicate question. There is no way a priori of knowing whether convergence happens for all such  $z$ , no such  $z$ , or some  $z$ . This question will not be of primary importance to us, so if you are interested in learning more, consult the exercises in the text.

## 2. DIFFERENTIATING POWER SERIES

An important result in real calculus states that, within a power series' radius of convergence, a power series is differentiable, and its derivative can be obtained by differentiating the individual terms of the power series term-by-term. The same holds true for complex power series:

**Theorem 2.** *Let  $f(z) = \sum a_n z^n$  be a power series with radius of convergence  $R > 0$ . Then  $f$  is holomorphic for  $|z| < R$ , and its derivative is equal to the power series  $f'(z) = \sum n a_n z^{n-1}$ , obtained by differentiating  $f(z)$  term by term. Also,  $f'$  has the same radius of convergence as  $f$ .*

*Proof.* First, notice that  $\sum a_n z^n$  and  $\sum n a_n z^{n-1}$  have the same radius of convergence because  $\limsup |a_n|^{1/n} = \limsup |n a_n|^{1/n}$ . Indeed, one can prove using single variable calculus that  $\lim n^{1/n} = 1$ . Fix  $z_0$  in the disc of convergence, so that  $|z_0| < r < R$  for some value of  $r$ . Let  $S_N(z), E_N(z)$  be defined by  $S_N(z) = \sum_{n \leq N} a_n z^n$ ,  $E_N(z) = \sum_{n > N} a_n z^n$ . (You should think of  $S_N$ , which consists of the lower order terms of the power series, as a 'main term', and  $E_N$  as an 'error term'.) Let  $g(z) = \sum n a_n z^{n-1}$

be the term-by-term derivative of  $f$ . We want to prove that  $f'(z_0) = g(z_0)$ , which means that we want to show

$$\lim_{h \rightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) = 0.$$

Rigorously, this means that given any  $\varepsilon > 0$ , we must show there exists a  $\delta > 0$  such that if  $0 < |h| < \delta$ , then the expression above has absolute value  $< \varepsilon$ . To do so, we will break the expression into three parts and estimate each of those separately using some sort of information in the hypotheses of the theorem. More precisely, since  $f(z) = S_N(z) + E_N(z)$ , we will write

$$\begin{aligned} & \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) = \\ & \left( \frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0) \right) + (S'_N(z_0) - g(z_0)) + \frac{E_N(z_0 + h) - E_N(z_0)}{h}. \end{aligned}$$

How does this help? Let's analyze each term one-by-one. The first term is

$$\frac{S_N(z_0 + h) - S_N(z_0)}{h} - S'_N(z_0).$$

Since  $S_N$  is just a polynomial, its derivative is  $S'_N$ . Therefore this term approaches 0 as  $h \rightarrow 0$ . In other words, given  $\varepsilon/3 > 0$ , we can find  $\delta > 0$  such that  $0 < |h| < \delta$  implies  $\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \varepsilon/3$ .

The second term is  $S'_N(z_0) - g(z_0)$ . Since  $S'_N(z_0) \rightarrow g(z_0)$  as  $N \rightarrow \infty$  (because we know that  $g(z)$  is a power series which converges absolutely for  $|z| < R$ , and  $S'_N(z)$  is the  $N$ th partial sum of this power series), this means that for any  $\varepsilon/3 > 0$ , we can find some  $N_1$  such that if  $N > N_1$ , then  $|S'_N(z_0) - g(z_0)| < \varepsilon/3$ .

The third term is the most tricky to estimate effectively. We can write  $E_N(z_0 + h) - E_N(z_0) = \sum_{n>N} a_n(z_0 + h)^n - a_n z_0^n$ . We use the algebraic identity  $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$ :

$$(z_0 + h)^n - z_0^n = h((z_0 + h)^{n-1} + (z_0 + h)^{n-2}z_0 + \dots + z_0^{n-1}).$$

Therefore

$$\frac{E_N(z_0 + h) - E_N(z_0)}{h} = \sum_{n>N} a_n((z_0 + h)^{n-1} + (z_0 + h)^{n-2}z_0 + \dots + z_0^{n-1}).$$

Notice that for  $h$  sufficiently small,  $|z_0 + h| < r$  as well as  $|z_0| < r$ . Therefore, if we stick absolute values on everything in the above expression and apply the triangle inequality, we obtain

$$\left| \frac{E_N(z_0 + h) - E_N(z_0)}{h} \right| \leq \sum_{n>N} |a_n| n r^{n-1}.$$

However, notice that the series on the right converges, and furthermore, its value approaches 0 as  $N \rightarrow \infty$ . Indeed, notice that  $\sum_{n>N} |a_n| n r^{n-1}$  is just the end of the series  $g(r)$  with absolute values on all of its individual terms, and we know that  $g(z)$  converges absolutely for  $|z| < R$ . So the series in question does converge, and

since it is the end of a convergent series, it must approach 0 as  $N \rightarrow \infty$ . Therefore, given some  $\varepsilon/3 > 0$ , we can find an  $N_2$  such that for all small  $h$  and  $N > N_2$ ,  $\left| \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| < \varepsilon/3$ .

Now select any  $N > \max(N_1, N_2)$ . Then select  $\delta > 0$  such that the first term  $\left| \frac{S_N(z_0+h) - S_N(z_0)}{h} - S'_N(z_0) \right| < \varepsilon/3$  for  $|h| < \delta$ . Then an application of the triangle inequality shows that

$$\left| \left( \frac{S_N(z_0+h) - S_N(z_0)}{h} - S'_N(z_0) \right) + (S'_N(z_0) - g(z_0)) + \frac{E_N(z_0+h) - E_N(z_0)}{h} \right| < \varepsilon$$

for  $|h| < \delta$ , as desired.  $\square$

We can apply the above theorem  $k$  times to obtain the following corollary:

**Corollary 1.** *A power series is infinitely (complex) differentiable in its disc of convergence, and each of its  $k$ th derivatives can be obtained by differentiating term-by-term  $k$  times. The resulting power series has radius of convergence equal to the original power series.*

The next corollary shows how the values of these derivatives at 0 are related to the coefficients of the power series.

**Corollary 2.** *Let  $f(z) = \sum a_n z^n$  be a power series with radius of convergence  $R > 0$ . Then  $a_k = \frac{f^{(k)}(0)}{k!}$ .*

*Proof.* By the previous corollary we know that  $f$  is infinitely differentiable at 0. Furthermore, we know that

$$f^{(k)}(z) = \sum_{n \geq k} a_n n(n-1)(n-2) \dots (n-k+1) z^{n-k},$$

by term-by-term differentiating  $k$  times. Plugging in  $z = 0$  gives

$$f^{(k)}(0) = a_k k(k-1) \dots 2 \cdot 1 = a_k k!,$$

since all terms with  $z^{n-k}$  where  $n > k$  vanish at  $z = 0$ . Therefore  $a_k = \frac{f^{(k)}(0)}{k!}$ , as desired.  $\square$

### 3. ANALYTIC FUNCTIONS

Let  $f(z)$  be a complex function. If there exists a power series  $\sum a_n (z - z_0)^n$  with radius of convergence  $R > 0$  which equals  $f(z)$  for all  $z$  in an open disc centered at  $z_0$ , then we say that  $f(z)$  is *analytic* at  $z_0$ : in other words, near  $z_0$ ,  $f$  is equal to a power series centered at  $z_0$  which converges in a non-trivial region.

Being analytic is a strong property: if  $f$  is analytic at  $z$ , then we know  $f$  can be locally expressed using a power series near  $z$ , which means all the properties of power series will transfer to  $f$  near  $z$ . Right now, we have very few tools to prove whether a function is analytic or not, but we will soon see that virtually all holomorphic functions are also analytic (a truly remarkable fact! One of the exercises from the book gives an example of a  $C^\infty$  real function which is not analytic.) In real calculus, one proves that a Taylor series for a function converges to its function using the Taylor remainder formula.

A basic fact which is useful (see Exercise 18 in the text) says that if  $f(z)$  is a power series with disc of convergence  $D$ , then  $f$  is analytic at every point of  $D$ .

### Examples.

- $e^z$  is analytic at  $z = 0$ . Indeed, because we define  $e^z$  to equal a power series which converges for all  $\mathbb{C}$ ,  $e^z$  is definitely equal to a power series in a neighborhood of 0. Furthermore, since the power series defining  $e^z$  converges for all  $z \in \mathbb{C}$ ,  $e^z$  is actually analytic on all of  $\mathbb{C}$ . (Notice that the power series expansion for  $e^z$  centered at a point  $z_0 \neq 0$  will not be  $1 + z + z^2/2! + \dots$ , which is a power series expansion centered around  $z = 0$ .)
- Any polynomial is analytic on all of  $\mathbb{C}$ . Indeed, a polynomial is its own power series centered at  $z = 0$ , which converges for all  $z \in \mathbb{C}$  (since there are only finitely many terms).
- If  $f, g$  are both analytic at  $z$ , then  $f + g, fg, f/g$  are all analytic at  $z$  ( $f/g$  only if  $g(z) \neq 0$ ). Compositions of analytic functions are also analytic.

#### 4. A DIGRESSION: DEFINING $e^z$ AND PROVING ITS PROPERTIES

As an illustration of some of the ideas in this section, we prove some of the basic properties of  $e^z$ . We will define  $e^z$  to be the value of the power series  $1 + z + z^2/2! + z^3/3! + \dots$ , which we know converges for all  $z \in \mathbb{C}$ .

The first basic property we want to prove is that  $(e^z)' = e^z$ . Indeed, if we take the power series expression for  $e^z$  and then differentiate it term-by-term, we find

$$(e^z)' = 1 + \frac{2 \cdot z}{2!} + \frac{3 \cdot z^2}{3!} + \dots = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = e^z.$$

Also, we want to show that  $e^0 = 1$ , but this is obvious by plugging in  $z = 0$  into the power series expression for  $e^z$ .

A little more tricky to show is the property that  $e^{z+w} = e^z e^w$  for all complex  $z, w$ . We begin by showing that if  $f(z) = \sum a_n z^n$  is a power series which satisfies  $f'(z) = f(z)$ , then  $f$  is a constant multiple of  $e^z$ . Indeed, term-by-term differentiation of  $f(z)$  yields

$$f'(z) = a_1 + 2a_2 z + 3a_3 z^2 + \dots = f(z) = a_0 + a_1 z + a_2 z^2 + \dots$$

From an exercise which you will solve on this week's homework, the only way two power series about the same center are equal in an open set containing their common center is if the corresponding coefficients of each  $z^n$  term are the same. Therefore, we see that  $a_0 = a_1, a_1 = 2a_2, a_2 = 3a_3, \dots$ . In particular, one can show that  $a_n = a_0/n!$ , which means that  $f(z) = a_0 e^z$ , as desired.

Consider  $e^{z+w}$  as a function of  $z$ , where we think of  $w$  as a fixed complex number. Then  $\frac{d}{dz} e^{z+w} = e^{z+w}$  by the chain rule and the fact that we know  $(e^z)' = e^z$ . Notice that  $e^{z+w}$  is analytic on all of  $\mathbb{C}$ , since  $e^z$  is. Therefore,  $e^{z+w}$  converges to its power series expansion about  $z = 0$ , and we have proven that a power series satisfies the differential equation  $f'(z) = f(z)$  if and only if  $f(z) = ce^z$  for some constant  $c$ . This means that  $e^{z+w} = ce^z$  for some complex number  $c$ . Plug in  $z = 0$  to see that  $e^w = c$ , so that  $e^{z+w} = e^w e^z$ , as desired.

We can then define  $\cos z, \sin z$  either by using their respective power series, or by declaring

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2}.$$