

THE CAUCHY INTEGRAL FORMULA

CONTENTS

1. The Cauchy Integral Formula	1
2. Consequences of the Cauchy Integral Formula	4
3. Holomorphic functions and their power series expansions	8
4. Uniqueness theorems for holomorphic functions	10

1. THE CAUCHY INTEGRAL FORMULA

We will now use Cauchy's Theorem for discs, keyhole contours, etc., to prove the first in a series of striking theorems and corollaries which show how different holomorphic functions are from their real counterparts. The first of these theorems is called the Cauchy Integral Formula, and relates the value of a holomorphic function at a point to the values of that function on a circle surrounding that point.

Theorem 1 (Cauchy Integral Formula). *Let C be a circle in the complex plane with positive orientation. Suppose f is holomorphic on an open set containing C and its interior. Let z be any point in the interior of C . Then*

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Before giving the proof of this theorem, we give some remarks and examples.

- This theorem is sometimes called a 'representation theorem', because it relates the value of a function satisfying certain properties (in this case, holomorphic) at a certain point to the value of the function at other points. As we will shortly see, this theorem is incredibly powerful and will be the starting point for a series of remarkable and unexpected theorems on holomorphic functions. More generally, in analysis, anytime a certain type of function satisfies some sort of representation theorem, it is likely that many more deep theorems will follow.
- Notice that what this theorem really says is that if f is holomorphic on a circle C and its interior, the values of f in the interior are *uniquely determined* by the values of f on C : after all, the integral in the Cauchy Integral Formula is over the curve C , and in the integrand we evaluate f on exactly C .
- For example, suppose f is holomorphic on C and its interior, and f is equal to 0 on C . Then the integrand in the Cauchy Integral Formula is 0 everywhere, so $f(z) = 0$ on the interior of C as well. This is the first of a series of results we will soon prove that show how holomorphic functions are uniquely determined by little information.

- This theorem is probably not like any theorem you have seen before. There is no analogue of this theorem for differentiable real valued functions. For example, if f were a differentiable real function (say, $\mathbb{R}^2 \rightarrow \mathbb{R}$), and if f is equal to 0 on a circle C , then the values of f on the interior are far from uniquely determined. Cauchy's Integral Theorem uses the fact that f is holomorphic, and not just real differentiable, in an essential way.
- The presence of the $2\pi i = \int_{S^1} 1/z dz$ is not an accident. As we will see in the proof, the basic strategy is to reduce the evaluation of the integral in the Cauchy Integral Formula to an integral around a circle centered at z .
- Suppose C is a circle which contains a complex number a in its interior. If we let $f(z) = 1$ everywhere, then the Cauchy Integral Formula applied to $z = a$ reads

$$1 = \frac{1}{2\pi i} \int_C \frac{1}{z - a} dz,$$

which basically is an integral you calculated on a previous homework assignment, with a considerable amount of work. This theorem provides a handy way of remembering such integrals.

- Recall that the real and imaginary parts of a holomorphic function are harmonic functions (they satisfy $u_{xx} + u_{yy} = 0, v_{xx} + v_{yy} = 0$). Then, with a fair amount of work, it is possible to look at the real and imaginary parts of the Cauchy Integral Formula and obtain an analogous representation theorem for harmonic functions! This theorem would hardly be obvious from the definition of harmonic functions, and shows how sometimes problems which originally seem like real analysis problems are often best approached via complex analysis. See Exercises 11 and 12 of Chapter 2 of the textbook for more information.

Proof. To prove the Cauchy Integral Formula, we begin by considering a keyhole contour $\Gamma_{\delta,\varepsilon}$ whose outer circle is C and whose inner circle is a small circle enclosing z of radius ε . The small number δ is the width of the keyhole corridor.

Notice that $f(\zeta)/(\zeta - z)$ (thought of as a function of ζ) is holomorphic at all points of C and its interior except at z , and we chose $\Gamma_{\delta,\varepsilon}$ to avoid z . Therefore $f(\zeta)/(\zeta - z)$ is holomorphic on an open set containing the keyhole contour $\Gamma_{\delta,\varepsilon}$ and its interior, so we can apply Cauchy's Theorem:

$$\int_{\Gamma_{\delta,\varepsilon}} \frac{f(\zeta)}{\zeta - z} d\zeta = 0.$$

Consider what happens as we let $\delta \rightarrow 0$, keeping ε fixed. As the corridor narrows, we are left with the outer circle C and the inner circle, which we will call C_ε , which has negative orientation. Because f is continuous on the corridor, as $\delta \rightarrow 0$ the integral along the two edges of the corridors will cancel each other out. (You will prove this on the next homework assignment.) Therefore, we have

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = - \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

(Remember that C_ε has negative orientation.) At this point we will rewrite the integrand in a slightly funny way:

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta) - f(z)}{\zeta - z} + \frac{f(z)}{\zeta - z}.$$

Notice that as we let $\zeta \in C_\varepsilon$, $(f(\zeta) - f(z))/(\zeta - z)$ is bounded: indeed, notice that this is just the expression that appears in the limit definition of $f'(z)$, as we let $\zeta \rightarrow z$: let $\zeta = z + h$, and this fraction becomes

$$\frac{f(\zeta) - f(z)}{\zeta - z} = \frac{f(z + h) - f(z)}{h}.$$

Since f is holomorphic at z , the limit of this fraction exists as $h \rightarrow 0$, so in particular for small ε , the absolute value of $(f(\zeta) - f(z))/(\zeta - z)$ is bounded by a number slightly larger than $|f'(z)|$.

As for the other part of the expression, notice

$$\int_{C_\varepsilon} \frac{f(z)}{\zeta - z} d\zeta = f(z) \int_{C_\varepsilon} \frac{1}{\zeta - z} d\zeta = -2\pi i f(z),$$

because C_ε is just a circle centered at z , with negative orientation. In summary, we have shown that

$$\begin{aligned} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta &= - \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - z} d\zeta \\ &= - \int_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta - \int_{C_\varepsilon} \frac{f(z)}{\zeta - z} d\zeta \\ &= - \int_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta + 2\pi i f(z) \end{aligned}$$

Let M be an upper bound for $|(f(\zeta) - f(z))/(\zeta - z)|$ for ζ close to z (ie, for ε small). Then

$$\left| \int_{C_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta \right| \leq 2\pi\varepsilon M \rightarrow 0, \text{ as } \varepsilon \rightarrow 0,$$

so we see that by taking $\varepsilon \rightarrow 0$,

$$\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = 2\pi i f(z).$$

Dividing both sides by $2\pi i$ proves the Cauchy Integral Formula. \square

Example. Let C be the circle $|z| = 2$ with positive orientation. Evaluate $\int_C \frac{1}{z^2 + 1} dz$.

Instead of direct evaluation, we will use the Cauchy Integral Formula together with partial fraction decomposition. Later on, we will see yet another (related) way to evaluate integrals like this.

First, notice that

$$\frac{1}{z^2 + 1} = \frac{1}{(z + i)(z - i)} = \frac{i}{2} \left(\frac{1}{z + i} - \frac{1}{z - i} \right).$$

Indeed, if we let

$$\frac{1}{z^2 + 1} = \frac{A}{z + i} + \frac{B}{z - i},$$

where A, B are complex numbers, then it is not hard to solve for A, B and find that $A = -B = i/2$. Therefore

$$\int_C \frac{1}{z^2 + 1} dz = \frac{i}{2} \left(\int_C \frac{1}{z + i} dz - \int_C \frac{1}{z - i} dz \right).$$

However, notice that since C encloses both $z = i, z = -i$, we know that

$$\int_C \frac{1}{z + i} dz = \int_C \frac{1}{z - i} dz = 2\pi i.$$

Therefore the original integral equals 0.

2. CONSEQUENCES OF THE CAUCHY INTEGRAL FORMULA

The first consequence of the Cauchy Integral Formula we discuss is also one of the most striking. It says that a holomorphic function on an open set Ω is not just once (complex) differentiable there, but infinitely complex differentiable on Ω !

Corollary 1 (Cauchy Integral Formula, higher derivatives). *Suppose f is holomorphic on an open set Ω which contains a positively oriented circle C and its interior. Suppose z is in the interior of C . Then f is n times complex differentiable at z , and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Proof. The strategy of the proof is to use induction. We have already proven the result for $n = 0$, since then the above formula is just the original Cauchy Integral Formula. Suppose we have already proven the result for $n - 1$, so that we know

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^n} d\zeta.$$

We want to prove this result for n . To do so, we construct the difference quotient for $f^{(n-1)}(z)$ that appears in the limit definition of the derivative and ask what happens as $h \rightarrow 0$: that is, consider

$$\lim_{h \rightarrow 0} \frac{f^{(n-1)}(z + h) - f^{(n-1)}(z)}{h}.$$

We want to show that this limit exists and is equal to the expression in the statement of the corollary. to do so, we use the inductive hypothesis on $f^{(n-1)}$ to see that

$$\frac{f^{(n-1)}(z + h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{h} \left(\frac{1}{(\zeta - (z + h))^n} - \frac{1}{(\zeta - z)^n} \right) d\zeta.$$

Let $A = \zeta - (z + h)$, $B = \zeta - z$ (this notation is slightly different from the book). Notice that

$$\begin{aligned} \frac{1}{A^n} - \frac{1}{B^n} &= \frac{B^n - A^n}{A^n B^n} = \frac{(B - A)(B^{n-1} + B^{n-2}A + \dots + A^{n-1})}{A^n B^n} \\ &= (B - A) \left(\frac{1}{A^n B} + \frac{1}{A^{n-1} B^2} + \dots + \frac{1}{B^n} \right). \end{aligned}$$

Also, notice that $B - A = h$. Therefore,

$$\frac{1}{(\zeta - (z + h))^n} - \frac{1}{(\zeta - z)^n} = h \left(\frac{1}{(\zeta - (z + h))^n (\zeta - z)} + \dots + \frac{1}{(\zeta - (z + h))(\zeta - z)^n} \right).$$

Therefore,

$$\begin{aligned} \frac{f^{(n-1)}(z + h) - f^{(n-1)}(z)}{h} &= \frac{(n-1)!}{2\pi i} \int_C \frac{f(\zeta)}{h} \left(\frac{1}{(\zeta - (z + h))^n} - \frac{1}{(\zeta - z)^n} \right) d\zeta = \\ &= \frac{(n-1)!}{2\pi i} \int_C f(\zeta) \left(\frac{1}{(\zeta - (z + h))^n (\zeta - z)} + \dots + \frac{1}{(\zeta - (z + h))(\zeta - z)^n} \right) d\zeta. \end{aligned}$$

To conclude the proof, we take $h \rightarrow 0$. Notice that we need to show that we can bring the $h \rightarrow 0$ limit into the integral over C . Once we do this, we see that

$$\lim_{h \rightarrow 0} \frac{f^{(n-1)}(z + h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \int_C f(\zeta) \frac{n}{(\zeta - z)^{n+1}} d\zeta,$$

as desired.

The interchange of limit and integral boils down to showing that the integrand uniformly converges as $h \rightarrow 0$. (Even though we only talked about sequences of functions converging as $n \rightarrow \infty$, all the same ideas apply if we let $h \rightarrow 0$. If you want, you can fill in the details as an exercise.) This is indeed the case, for example by the Weierstrass M-test, because if we let d be slightly less than the minimal distance of z from C , we can always choose d to be positive (since z is not on C), and then the integrand is bounded from above uniformly by Mn/d^{n+1} as $h \rightarrow 0$, where M is the maximum of f on C . □

Already this fact is remarkable – there is no analogue for real differentiable functions. Also notice that, because the real and imaginary parts of a holomorphic function are harmonic, it must also be true that a harmonic function is C^∞ ! This is yet another striking fact about harmonic functions which is best appreciated through complex analysis.

An easy application of the ML-lemma to the higher derivatives version of the Cauchy Integral Formula yields the following, which are frequently called the Cauchy Inequalities:

Corollary 2 (Cauchy Inequalities). *Suppose f is holomorphic on an open set containing a circle C and its interior, and let z be in the interior of C . Let $\|f\|_C = \sup_{\zeta \in C} |f(\zeta)|$ be the maximum size of f on C , and let R be the radius of C . Then*

$$|f^{(n)}(z)| \leq \frac{n! \|f\|_C}{R^n}.$$

Proof. Cauchy's Integral Formula reads

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Apply the ML-lemma, with $M = \|f\|_C/R^{n+1}$, $L = 2\pi R$ to prove Cauchy's Inequalities. \square

To show just how non-trivial all the mathematics we have done so far is, we now give a simple proof (perhaps the simplest) of the Fundamental Theorem of Algebra, which is a simple consequence of yet another striking fact about entire functions:

Corollary 3 (Liouville's Theorem). *Suppose f is an entire function. If f is bounded on \mathbb{C} , then f is constant.*

Proof. It suffices to show that $f' = 0$ on all of \mathbb{C} . To see this, apply Cauchy's Inequalities in the $n = 1$ case to find that

$$|f'(z)| \leq \frac{\|f\|_C}{R},$$

where $z \in \mathbb{C}$, and C is any circle of radius R centered at z . Suppose f is bounded from above by B ; ie, $|f(z)| \leq B$ for all of \mathbb{C} . Then

$$|f'(z)| \leq \frac{B}{R}.$$

Let $R \rightarrow \infty$ to see that $f'(z) = 0$. Since z was arbitrary, this proves the theorem. \square

For example, consider a function like $f(z) = \sin z$. We know how to define this for all of \mathbb{C} (via a power series expansion, say), and we know that $\sin z$ is entire. When restricted to real numbers, $\sin x$ is analytic everywhere but bounded by 1. This does not happen when \sin is extended to \mathbb{C} ! In particular, if we approach ∞ via non-real directions (say, via the imaginary axis), then $\sin z$ becomes unbounded.

Corollary 4 (The Fundamental Theorem of Algebra). *Suppose $f(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$ is a polynomial with complex coefficients. If $f(z)$ has no roots in \mathbb{C} , then f is constant. In other words, any non-constant complex polynomial has a complex root in \mathbb{C} .*

Proof. Suppose f has no complex roots. Then $1/f$ is entire. Furthermore, because $|f(z)| \rightarrow \infty$ as $z \rightarrow \infty$ (in any direction), this means that $1/f$ is bounded on \mathbb{C} . More precisely, given any positive number M , we can find a N such that if $|z| > N$, then $|f(z)| > M$. Indeed, suppose $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$. The triangle inequality implies that $|f(z)| \geq |a_n| |z|^n - |a_{n-1} z^{n-1} + \dots + a_0|$. If we let $a = \max(|a_{n-1}|, |a_{n-2}|, \dots, |a_0|)$, then by the triangle inequality

$$|a_{n-1} z^{n-1} + \dots + a_0| \leq na |z|^{n-1},$$

at least when $|z| > 1$. But notice that as $|z| \rightarrow \infty$, $|a_n z^n| - na |z|^{n-1} \rightarrow \infty$ as well. This proves the claim.

In summary, we know that there exists some N such that if $|z| > N$, then $|1/f(z)| < 1$, say. Also, on the closed disc $|z| \leq N$, $1/f(z)$ is continuous, so it achieves some maximum value (by the Extreme Value Theorem), so on all of \mathbb{C} $1/f(z)$ is bounded. Therefore Liouville's Theorem implies that $1/f$ is constant, hence f is constant. \square

Corollary 5. *Any polynomial $f(z) = a_n z^n + \dots + a_0$ with complex coefficients of degree n can be factored as*

$$f(z) = a_n(z - r_1)(z - r_2) \dots (z - r_n),$$

where the r_i are complex numbers (not necessarily distinct).

Proof. In general, regardless of whether or not $z - r$ divides a polynomial $p(z)$, we may write

$$p(z) = (z - r)q(z) + c,$$

where $q(z)$ is a polynomial of degree one less than $p(z)$, and c is a complex number. Indeed, this is basically a consequence of polynomial long division (more rigorously, this is a Euclidean division in complex polynomials, which one can show is possible in an algebra class). If we evaluate this equation at $z = r$, we see that if $p(r) = 0$, then $c = 0$. In other words, if r is a root of $p(z)$, then $(z - r)$ divides $p(z)$.

Therefore, given a degree n polynomial $f(z)$, the fundamental theorem of algebra says there is some complex root, say r_1 , of $f(z)$, and by what we have just shown $f(z) = (z - r_1)f_1(z)$. Repeatedly apply the same idea to $f_1(z)$ to find that $f(z) = (z - r_1)(z - r_2) \dots (z - r_n)c$, for some complex number c . Comparing leading coefficients, we find $c = a_n$. \square

Example. As an variant of this corollary, we show that any polynomial with real coefficients can be factored into a product of either linear or quadratic real polynomials. Recall that if $p(z)$ is a polynomial with real coefficients, and r is a root of $p(z)$, then \bar{r} is a root of $p(z)$ as well: indeed, using the fact that $p(z)$ has real coefficients, one finds that $\overline{p(r)} = p(\bar{r}) = 0$.

The Fundamental Theorem of Algebra guarantees that $p(z)$ has a complex root, say r_1 . If r_1 is real, then $p(z) = (z - r_1)q(z)$, and it is not hard to check that $q(z)$ is actually a polynomial with real coefficients. (Indeed, Euclidean divisions are possible for polynomials with real coefficients as well as complex coefficients.) If r_1 is not real, then \bar{r}_1 is also a root of $p(z)$, and we can write $p(z) = (z - r_1)(z - \bar{r}_1)q(z)$, where $q(z)$ is a polynomial of degree $n - 2$. First, notice that $(z - r_1)(z - \bar{r}_1) = z^2 - (r_1 + \bar{r}_1)z + |r_1|^2$ is a quadratic polynomial with real coefficients. Second, one can show that $q(z)$ has real coefficients, again essentially by applying a Euclidean division. (This time, the remainder will be a linear polynomial with real coefficients which has roots r_1, \bar{r}_1 , hence must be the 0 polynomial.)

Therefore we have shown that we can factor out a linear or quadratic real polynomial out of an arbitrary real polynomial, and what is left is still a real polynomial. Apply this repeatedly to obtain the desired conclusion.

3. HOLOMORPHIC FUNCTIONS AND THEIR POWER SERIES EXPANSIONS

Another consequence of the Cauchy Integral Formula is that functions which are holomorphic on open sets are actually analytic (ie, locally equal to a power series expansion) as well:

Corollary 6. *Suppose $f(z)$ is holomorphic on an open set Ω , and suppose z_0 is in Ω . Let D be an open disc centered at z_0 whose closure is entirely contained in Ω . Then there exists a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

which converges and is equal to $f(z)$ on all of D , and the coefficients are given by the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

Proof. Let $z \in D$ be any point in D , and let C be the boundary of D , so that z lies in the interior of C . Then the Cauchy Integral Formula says that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

The idea is to write $1/(\zeta - z)$ as the value of a (uniformly) converging geometric series on C . Indeed,

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{\frac{1}{\zeta - z_0}}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \left(1 + \left(\frac{z - z_0}{\zeta - z_0} \right) + \left(\frac{z - z_0}{\zeta - z_0} \right)^2 + \dots \right).$$

Notice that since $\zeta \in C$, $|z - z_0|/|\zeta - z_0| = r < 1$, because $|\zeta - z_0|$ is the radius of C , while $|z - z_0|$ is less than the radius of C , since z is in the interior of C . Therefore, we can rewrite the Cauchy Integral Formula as

$$f(z) = \frac{1}{2\pi i} \int_C f(\zeta) \frac{1}{\zeta - z_0} \left(1 + \left(\frac{z - z_0}{\zeta - z_0} \right) + \left(\frac{z - z_0}{\zeta - z_0} \right)^2 + \dots \right) d\zeta.$$

Notice that the integrand is a uniformly converging series on C , because of the Weierstrass M-test, for example. Therefore we can interchange the infinite sum and the integral:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} d\zeta + \frac{1}{2\pi i} \int_C \frac{f(\zeta)(z - z_0)}{(\zeta - z_0)^2} d\zeta + \frac{1}{2\pi i} \int_C \frac{f(\zeta)(z - z_0)^2}{(\zeta - z_0)^3} d\zeta + \dots$$

Notice that we can factor the powers of $(z - z_0)$ out of each integral since they are just constant inside the integral (since ζ is the variable we integrate with respect to). Also, the Cauchy Integral Formulas also tell us that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta.$$

Therefore,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

This is true for any $z \in D$, so this proves the corollary. \square

This corollary not only tells us that if f is holomorphic in an open set containing z , then f is analytic at z (ie, is equal to a power series in a neighborhood), but also that this power series expansion is valid in ‘as large a disc as is possible’. More specifically, a power series representation for an analytic function is valid as long as that function remains holomorphic.

Examples.

- Consider an entire function, like e^{z^2+z} . Then this function has a power series expansion around $z_0 = 0$ (or any other point), and this power series expansion converges for all z , and this function is equal to this power series on all of \mathbb{C} .
- Consider $f(z) = \frac{1}{z^2+9}$. This function is holomorphic everywhere except at $z = \pm 3i$, where $|f(z)| \rightarrow \infty$ as $z \rightarrow \pm 3i$. Then this function has a power series expansion around $z = 0$ (and one could easily compute it) whose radius of convergence is $R = 3$. Indeed, given any $R' < 3$, this power series $g(z) = \sum a_n z^n$ converges for $|z| < R'$, because $f(z)$ is holomorphic in an open set containing $|z| \leq R'$. On the other hand, we must also have $R \leq 3$, because if $R > 3$, then $g(z)$ would be bounded as $z \rightarrow 3i$ (since $g(3)$ converges). However, this is impossible, because $g(z) = f(z)$ for all $|z| < 3$, and $|f(z)| \rightarrow \infty$ as $z \rightarrow 3i$ from below.
- In a similar vein, consider the function $f(z) = 1/\sin(\pi z)$. What is the radius of convergence of the power series expansion of $f(z)$ around $z_0 = 2 + 4i$?

First, we determine the set of z for which $f(z)$ is holomorphic. Since $\sin(\pi z)$ is entire, this is the same as determining when $\sin(\pi z) = 0$. Recall the identity

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i},$$

which for example can be deduced by looking at the power series expansions of $\sin z, e^{iz}, e^{-iz}$. This holds true for all $z \in \mathbb{C}$; hence, to solve $\sin z = 0$ is the same as solving $e^{iz} = e^{-iz}$. Let $z = x + yi$. Then we want to solve $e^{-y+xi} = e^{y-ix}$. In polar form, this is $e^{-y}e^{ix} = e^y e^{-ix}$. Therefore $e^{-y} = e^y$, which implies that $y = 0$. Also, $e^{ix} = e^{-ix}$ has solution if and only if $\cos x = \cos -x = \cos x, \sin x = \sin -x = -\sin x$, which evidently only has solutions when $\sin x = 0$, or when $x = n\pi$, where n is an integer. Therefore, $\sin z = 0$ if and only if $z = n\pi + 0i$. (Notice it is obvious that these are all solutions to $\sin z = 0$. The reason we had to perform calculations was to check that these were the only complex zeros of $\sin z$.)

Therefore $f(z) = 1/\sin(\pi z)$ is holomorphic whenever z is not an integer. Therefore, by similar reasoning to the above, the radius of convergence of the power series expansion of $f(z)$ around $z_0 = 2 + 4i$ is 4, because this is the radius of the largest open disc centered at $2 + 4i$ on which $f(z)$ is holomorphic, and $|f(z)| \rightarrow \infty$ as $z \rightarrow 2$.

- Notice that these ideas for determining the radius of convergence of power series of holomorphic functions do not apply to their real analogues. For example, $f(x) = 1/(x^2 + 1)$ is an analytic function on all of \mathbb{R} , but its power series expansion around $x = 0$ is $1 - x^2 + x^4 - x^6 + \dots$, which has radius of convergence $R = 1$, not ∞ . Of course, the reason for this is because a power series is genuinely a complex analytic object, and the fact that $R = 1$ is reflected by the fact that $f(x)$ cannot be holomorphically extended to an open disc containing $z = \pm i$, which are distance 1 from the origin.

4. UNIQUENESS THEOREMS FOR HOLOMORPHIC FUNCTIONS

Since holomorphic functions on open sets are locally equal to power series, we might expect that certain properties of power series transfer over to holomorphic functions. In particular, recall we found out that power series were uniquely determined by their values on either an open disc centered at the center of the disc of convergence, or by their values on a sequence of points converging to the center of the disc of convergence. Something more general is true for holomorphic functions.

Theorem 2. *Suppose $f(z)$ and $g(z)$ are holomorphic on an open connected set Ω , and that z_n is a sequence of points in Ω converging to some limit in Ω with $f(z_n) = g(z_n)$, with no z_n equal to this limit. Then $f = g$ on all of Ω .*

Proof. Let $w = \lim z_n$. Since $w \in \Omega$, we can expand power series (whose radii of convergence are > 0) for f, g around w ; call them $\sum a_n(z - w)^n, \sum b_n(z - w)^n$. Then by an earlier homework problem, these power series are equal to each other, because they are equal on a sequence of points converging to their center w (namely, the sequence z_n). This means that $f = g$ on an open disc centered at w – namely, the smaller of the discs of convergence for their two power series representations at w .

Given that $f = g$ on an open disc in Ω , we now show that $f = g$ for all of Ω . To do this, we use the fact that Ω is connected. Let U be the interior of the set of points where $f = g$ in Ω . Then by definition U is open. Also, U is non-empty since it contains the open disc where $f = g$. Finally, U is also closed in Ω , because given any sequence of points in U converging to a limit z in Ω , continuity of f, g at z implies that $f(z) = g(z)$. Then the previous argument also shows that there is an open disc centered at z on which f, g are equal, so that z is an interior point of U .

Therefore, if we let V be the complement of U in Ω , we have a disjoint union $\Omega = U \cup V$, where U is both open and closed, as well as non-empty. Because Ω is connected, this implies that $U = \Omega$, as desired. \square

Corollary 7. *If $f(z)$ is holomorphic on a region Ω , and z_n is a sequence of points with limit $z \in \Omega$ such that $f(z_n) = 0$, then $f = 0$ on all of Ω . Similarly, if $f(z) = 0$ on any open disc in Ω , then $f(z) = 0$ on all of Ω .*

Proof. This is just a routine application of the previous theorem: let $g(z) = 0$ everywhere. The second part is also clear, because if $f(z) = 0$ on an open disc, then select any sequence converging to the center of that disc (or any other point of the disc) and apply the previous part. \square

Suppose f is holomorphic on a region Ω , and F is holomorphic on a bigger region Ω' containing Ω such that $F = f$ on Ω . Then we call F an *analytic continuation* of

f to Ω' . The previous results imply that analytic continuations to larger regions are unique:

Corollary 8. *Suppose F is an analytic continuation of f from a region Ω to a region Ω' . Then F is unique.*

Proof. If F_1, F_2 are analytic continuations of f from Ω to Ω' , then $F_1 = F_2$ on an open disc contained in Ω' (namely, any open disc contained in Ω). The previous results then imply that $F_1 = F_2$ on all of Ω' . \square

- The uniqueness of an analytic continuation is peculiar. It implies that a holomorphic function is uniquely determined by a very small amount of information – essentially, the values of a holomorphic function on an infinite sequence of distinct points converging to a limit is enough to completely determine that function. For example, suppose you were told that $f(z) = 1/z$ on a small open disc centered at $z = 100$, and $f(z)$ were a function holomorphic on a region Ω . Then $f(z) = 1/z$ on all of Ω . In particular, the fact that $f(z) = 1/z$ far away from the origin is already enough to preclude the possibility of extending f to a holomorphic function on a connected open set containing the origin. In other words, even though $f(z) = 1/z$ is holomorphic and well-behaved on small open discs far from the origin, that already contains enough information to specify that any analytic continuation must blow up to infinity near the origin!
- There is no analogue of uniqueness of continuation for real differentiable functions. For example, it is easy to write down two different real differentiable functions on \mathbb{R} which are equal on an open subset. In particular, it is possible to write down a real differentiable function which is not zero everywhere but equal to 0 on an open interval; for example, $f(x) = 0, x < 0, f(x) = x^2, x \geq 0$ is such a function.
- Another easy consequence of the above results is that if a holomorphic function on a region Ω is constant on some open disc contained in Ω , then f is constant on all of Ω ; indeed, if $f(z) = c$ on some open disc, apply the uniqueness results above to $f - c$, which equals 0 on that open disc, and therefore must be equal to 0 on all of Ω .