Equations of motion in General Three-Body Problem

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{y}_1 \\
\dot{z}_1 \\
\dot{x}_2 \\
\dot{y}_2 \\
\dot{z}_2 \\
\dot{x}_3 \\
\dot{y}_3 \\
\dot{z}_3
\end{bmatrix} = \begin{bmatrix}
\frac{\vec{p}_1}{m_1} \\
\frac{m_2}{l_1^2 (x_2 - x_1)(x_2 - x_1)} \dot{x}_1 - \frac{m_2}{l_1^2 (x_2 - x_1)} \dot{z}_1 \\
\frac{m_2}{l_1^2 (x_2 - x_1)} \dot{y}_1 + \frac{m_2}{l_1^2 (x_2 - x_1)} \dot{z}_1 \\
\frac{m_1}{l_1^2 (x_1 - x_2)(x_1 - x_2)} \dot{x}_2 - \frac{m_1}{l_1^2 (x_1 - x_2)} \dot{z}_2 \\
\frac{m_1}{l_1^2 (x_1 - x_2)} \dot{y}_2 + \frac{m_1}{l_1^2 (x_1 - x_2)} \dot{z}_2 \\
\frac{m_1}{l_1^2 (x_1 - x_2)} \dot{y}_2 + \frac{m_1}{l_1^2 (x_1 - x_2)} \dot{z}_2 \\
\frac{m_3}{l_2^2 (x_3 - x_2)(x_3 - x_2)} \dot{x}_3 - \frac{m_3}{l_2^2 (x_3 - x_2)} \dot{z}_3 \\
\frac{m_3}{l_2^2 (x_3 - x_2)} \dot{y}_3 + \frac{m_3}{l_2^2 (x_3 - x_2)} \dot{z}_3 \\
\frac{m_3}{l_2^2 (x_3 - x_2)} \dot{y}_3 + \frac{m_3}{l_2^2 (x_3 - x_2)} \dot{z}_3
\end{bmatrix}
\]
Appendix B

Derivation of the Lagrange - Jacobi identity

In the center of mass coordinate system, the moment of inertia of the three-body system is:

\[ I = M_1 v_1^2 + M_2 v_2^2 + M_3 v_3^2 \]

Or in the Lagrangian form:

\[ I = \frac{M_1 M_2 M_3}{M} \left( \frac{v_1^2}{M_1} + \frac{v_2^2}{M_2} + \frac{v_3^2}{M_3} \right) \]

where:

\[ M = M_1 + M_2 + M_3 \]

\[ r_{21} = ||\vec{r}_2 - \vec{r}_1|| \]

\[ r_{13} = ||\vec{r}_1 - \vec{r}_3|| \]

\[ r_{23} = ||\vec{r}_2 - \vec{r}_3|| \]

Differentiating \( I \) twice with respect to time gives:

\[ \dot{I} = 2m_1 \vec{r}_1 \cdot \ddot{\vec{r}}_1 + 2m_2 \vec{r}_2 \cdot \ddot{\vec{r}}_2 + 2m_3 \vec{r}_3 \cdot \ddot{\vec{r}}_3 \]

\[ \ddot{I} = 2m_1 (\ddot{r}_1 + \vec{r}_1 \cdot \dddot{r}_1) + 2m_2 (\ddot{r}_2 + \vec{r}_2 \cdot \dddot{r}_2) + 2m_3 (\ddot{r}_3 + \vec{r}_3 \cdot \dddot{r}_3) \]

\[ \frac{1}{2} \ddot{I} = 2T + V = 2E - V \]

\[ T = \frac{1}{2} \sum_{i=1}^{n} m_i v_i^2 \]

\[ E = T + V \]

\[ V = -G \sum_{j=1}^{n} \sum_{k=j+1}^{n} \frac{m_i M_j}{r_{jk}} \]

From Vallone's p32 - p33
Appendix C:

Translate Equations of Motion to Rotating Inertial Frame:

To go to the rotating inertial reference frame we wish to scale the problem to depend on a single parameter, \( \mu \).

- Mass of planet = \( M \)
- Mass of Sun = \( 1 - \mu \)

\[ \mu = \frac{\text{mass of planet}}{\text{total mass}} = \frac{M}{M + M} \]

- Distance between primaries (planet and Sun) = 1
  - Distance from center of mass to planet = \( 1 - \mu \)
  - Distance from center of mass to Sun = \( \mu \)

- Unit of time chosen so that mass moment of primaries is \( m = 1 \), hence angular velocity = time

- From these it follows that the gravitational constant, \( C = 1 \).

To describe the rotating inertial frame, we first start in the fixed inertial frame: \( \Xi, \mu \). In this frame the equations of motion are:

Sun:
\[
\begin{cases}
\dot{\xi} = -\mu \omega t \\
\dot{\eta} = -\omega t
\end{cases}
\]

Planet:
\[
\begin{cases}
\dot{\xi} = (1 - \mu) \omega t \\
\dot{\eta} = (1 - \mu) \omega t
\end{cases}
\]

But, we want to translate to the rotating synodic frame \( \xi_n \) where the primaries are stationary.
The Hamiltonian of the asteroid in the fixed frame is:

\[ H = \frac{1}{2} (p_1^2 + p_2^2) - \frac{1-u}{p_1} - \frac{u}{p_2} \]

where

\[ p_1 = (\varepsilon + u\cos\theta)^2 + (1 + u\sin\theta)^2 \]

\[ p_2 = (\varepsilon - (1-u)\cos\theta)^2 + (1 - (1-u)\sin\theta)^2 \]

The fixed frame \( \rightarrow \) rotating frame are related by the following equations:

\[ \varepsilon = \varepsilon \cos\theta - n\sin\theta \]

\[ u = \varepsilon \sin\theta + n\cos\theta \]

The transformation can be accomplished using a generating function:

\[ F = F(p_1, p_2, \varepsilon, u) \]

\[ = - (\varepsilon \cos\theta - n\sin\theta) p_2 - (\varepsilon \sin\theta + n\cos\theta) p_1 \]
The new momenta are:

\[ P_\varepsilon = - \frac{\partial F}{\partial \varepsilon} = p_\varepsilon \cos \varepsilon + p_n \sin \varepsilon \]

\[ P_n = - \frac{\partial F}{\partial n} = - p_\varepsilon \sin \varepsilon + p_n \cos \varepsilon \]

The new Hamiltonian is:

\[ H = H + \frac{\partial F}{\partial \varepsilon} = \frac{1}{2} (P_\varepsilon^2 + P_n^2) - \frac{1 - \mu}{\rho_1} - \frac{\mu}{\rho_2} + \varepsilon P_\varepsilon \sin \varepsilon + \mu P_n \cos \varepsilon + \varepsilon (-P_n) + \mu P_\varepsilon 
\]

where:

\[ \rho_1^2 = (\varepsilon + \mu)^2 + \mu^2 \]

\[ \rho_2^2 = (\varepsilon - (1 - \mu))^2 + \mu^2 \]

The equations of motion in the synodic frame are now:

\[ \dot{\varepsilon} = \frac{\partial H}{\partial P_\varepsilon} = P_\varepsilon + \mu \]

\[ \dot{\mu} = \frac{\partial H}{\partial P_n} = P_n - \varepsilon \]

\[ \dot{P}_\varepsilon = - \frac{\partial H}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon} \left( \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right) + P_n - \frac{\partial}{\partial \varepsilon} \left( \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right) + \varepsilon + \mu \]

\[ \dot{P}_n = - \frac{\partial H}{\partial n} = \frac{\partial}{\partial n} \left( \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right) - P_\varepsilon - \frac{\partial}{\partial n} \left( \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right) - \varepsilon + \mu \]

\[ \dot{\varepsilon} = \dot{\varepsilon} - \dot{\mu} = \dot{\mu} + \varepsilon + \frac{\partial}{\partial \varepsilon} \left( \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \]

\[ \dot{\mu} + \dot{\varepsilon} = -\dot{\varepsilon} + \dot{\mu} + \frac{\partial}{\partial n} \left( \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \]
These equations simplify to:

\[
\ddot{\theta} = 2\dot{\theta} + \dot{\theta} - (1-\mu)(\dot{\theta} + \mu)\left[ (\dot{\theta} + \mu)^2 - \mu^2 \right]^{-3/2} \\
- \mu \left[ (\dot{\theta} - (1-\mu))^2 + \mu^2 \right]^{-3/2}
\]

\[
\ddot{n} = -2\dot{\theta} + \dot{n} - \mu (1-\mu)\left[ (\dot{\theta} + \mu)^2 + \mu^2 \right]^{-3/2} \\
- \mu \left[ (\dot{\theta} - (1-\mu))^2 + \mu^2 \right]^{-3/2}
\]

These equations form the following system of coupled first order equations:

\[
\dot{\theta} = p \\
\dot{p} = 2q + \dot{\theta} - (1-\mu)(\dot{\theta} + \mu)\left[ (\dot{\theta} + \mu)^2 - \mu^2 \right]^{-3/2} \\
- \mu \left[ (\dot{\theta} - (1-\mu))^2 + \mu^2 \right]^{-3/2}
\]

\[
\dot{n} = q \\
\dot{q} = -2p + \dot{n} - \mu (1-\mu)\left[ (\dot{\theta} + \mu)^2 + \mu^2 \right]^{-3/2} \\
- \mu \left[ (\dot{\theta} - (1-\mu))^2 + \mu^2 \right]^{-3/2}
\]
Appendix D

Language Points: Locations and Stabilities

Locations:

Language points occur where \( \dot{\xi} = \dot{n} = 0 \)

This occurs when \( \frac{\partial \Omega}{\partial \xi} = \frac{\partial \Omega}{\partial n} = 0 \) where

\[
\Omega = \frac{1}{2} (\xi^2 + n^2) + \frac{1 - \mu}{\rho_1} + \frac{\mu}{\rho_2}
\]

\( \rho_1 \) and \( \rho_2 \) defined in Appendix C

because

\[
\ddot{\xi} - 2\dot{\xi} = \frac{\partial \Omega}{\partial \xi}
\]

and

\[
\ddot{n} + 2\dot{n} = \frac{\partial \Omega}{\partial n}
\]

Calculating these derivatives we get a pair of equations for the language equation

\[
\frac{\partial \Omega}{\partial \xi} = \frac{1 - \mu}{\rho_1^3} \xi - \mu \left( \xi - (1 - \mu) \right) = 0 \quad (5.17)
\]

\[
\frac{\partial \Omega}{\partial n} = \frac{1 - \mu}{\rho_1^3} n - \frac{\mu n}{\rho_2^3} = 0 \quad (5.18)
\]

The second equation has the trivial solution where \( n = 0 \). The first equation then gives:

\[
\xi - \frac{(1 - \mu)(\xi + \mu)}{[\xi + \mu]^2} \cdot \frac{\mu(\xi - (1 - \mu))}{\sqrt{\xi^2 - (1 - \mu)^2}}^\frac{3}{2} = 0
\]

This has the solution:
\[ \varepsilon - \frac{1 - \mu}{(\varepsilon + \mu)^2} - \frac{\mu}{(\varepsilon - (1 - \mu))^2} = 0 \quad 1 - \mu < \varepsilon \]

\[ \varepsilon - \frac{1 - \mu}{(\varepsilon + \mu)^2} + \frac{\mu}{(\varepsilon - (1 - \mu))^2} = 0 \quad -\mu < \varepsilon < 1 - \mu \]

\[ \varepsilon + \frac{1 - \mu}{(\varepsilon + \mu)^2} + \frac{\mu}{(\varepsilon - (1 - \mu))^2} = 0 \quad \varepsilon > -\mu \]

These can be solved numerically.

For \( n < 0 \) we have the second equation

\[ 1 - \frac{(1 - \mu)}{\rho_1^3} - \frac{\mu}{\rho_2^3} = 0 \quad (5.18) \]

Multiplying by \( \varepsilon + \mu \) and subtracting the first equation \((5.17)\)

\[ \frac{\mu}{\rho_2^3} - \mu = 0 \]

Therefore \( \rho_2 = 1 \)

Multiplying \((5.18)\) by \( \varepsilon - (1 - \mu) \) and subtracting \((5.17)\) we get

\[ 1 - \mu - \frac{1 - \mu}{\rho_1^3} = 0 \]

Therefore \( \rho_1 = 1 \)

Since \( \rho_1 = 1 \) and \( \rho_2 = 1 \) then the points are those on the equilateral triangle with the points of the primavera:

\[ (\varepsilon - (1 - \mu))^2 + \mu^2 = 1 \]

\[ (\varepsilon + \mu)^2 + \mu^2 = 1 \]

So \( \varepsilon = \frac{1}{2} - \mu \) \( , \mu = \pm \frac{1}{2} \)
We can explore the stability putting a body near a Lagrangian point. Let $(\xi_0, \eta_0)$ be any Lagrangian point as $(x, y)$ the position of the body relative to this point:

\[
\begin{align*}
\xi &= \xi_0 - x \\
y &= \eta_0 - y
\end{align*}
\]

Consequently in the neighborhood about the Lagrangian point we can linearly approximate the derivatives of $x$ and $y$:

\[
\frac{\partial x}{\partial \xi} = \frac{\partial y}{\partial \eta} = 0
\]

\[
\frac{\partial x}{\partial \xi} = \xi \frac{\partial^2 x}{\partial \xi^2} + y \frac{\partial^2 y}{\partial \xi \partial \eta}
\]

\[
\frac{\partial y}{\partial \eta} = -\xi \frac{\partial^2 x}{\partial \xi \partial \eta} + \eta \frac{\partial^2 y}{\partial \eta^2}
\]

So the equations of motion are:

\[
\begin{align*}
\ddot{x} - 2\gamma &= \xi \frac{\partial^2 x}{\partial \xi^2} + y \frac{\partial^2 y}{\partial \xi \partial \eta} \\
\ddot{y} + 2\kappa &= -\xi \frac{\partial^2 x}{\partial \xi \partial \eta} + \eta \frac{\partial^2 y}{\partial \eta^2}
\end{align*}
\]

where $\alpha = \frac{1-\mu}{\rho_1^2} + \frac{\mu}{\rho_2^2}$.

Considering the three Lagrangian Points, the derivatives of $x$ and $y$ are:

\[
\begin{align*}
\frac{\partial^2 x}{\partial \xi^2} &= 1 + 2\kappa \\
\frac{\partial^2 x}{\partial \eta^2} &= 1 - \alpha \\
\frac{\partial^2 y}{\partial \xi \partial \eta} &= 0
\end{align*}
\]

as $\eta = 0$ for $L_1, L_2$ and $L_3$. 

From Vallance p125-128
The equations of motion are now:
\[
\begin{align*}
\dot{x} &= 2iy = x(1 + 2\alpha) \\
\dot{y} &= 2\dot{x} = y(1 - \alpha)
\end{align*}
\]

We can study their trajectories to determine stability:
\[
\begin{align*}
x &= Ae^{ut} \\
y &= Be^{ut}
\end{align*}
\]
where A, B, and w are constants.

If \( \text{Re}(\omega) \neq 0 \), then the solutions grow without limit and are unstable.

If \( \text{Re}(\omega) = 0 \), then the solutions are stable.

Substituting in the solutions and performing some algebra we see:
\[
\begin{align*}
A\omega^2 - 2B\omega &= A(1 + 2\alpha) \\
B\omega^2 + 2A\omega &= B(1 - \alpha)
\end{align*}
\]

Eliminating A & B and performing more algebra we get:
\[
\omega^4 + \omega^2(2 - \alpha) + (1 + 2\alpha)(1 - \alpha) = 0
\]

If \( \text{Re}(\omega) = 0 \), then there must be two negative solutions for \( \omega^2 \).

The product of these roots must be positive, therefore:
\[
(1 + 2\alpha)(1 - \alpha) > 0
\]

Therefore: \( \alpha < 1 \)

But the irreversibility point must satisfy (5.17) from Appendix C
\[
\frac{d\rho}{d\tau} = \rho - \frac{2(1-\gamma)(3+\omega)}{\rho^3} - \frac{\rho(\gamma - (1-\gamma))}{\rho^2} = 0
\]

Rearranging the terms:
\[
\rho - \rho\left[\frac{1-\gamma}{\rho^2} + \frac{\gamma}{\rho^2}\right] - \frac{\rho(\gamma - (1-\gamma))}{\rho^2} = 0
\]
\[ \epsilon(1 - \mu) - \mu(1 - \mu) \left[ \frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right] = 0 \]

from which we see

\[ 1 - \mu = \frac{\mu(1 - \mu)}{\epsilon} \left[ \frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right] \]

Therefore for all Lagrangian points on \( \mathcal{E} \) axis, like \( L_1, L_2, L_3 \) we're considering, the bracketted expression and \( \mathcal{E} \) have opposite signs. The right hand side must, therefore, be negative, so:

\[ \mu > 1. \]

This contradicts our previous requirement. Therefore we cannot find purely imaginary solutions for \( \omega \) for \( L_1, L_2, L_3 \). Therefore \( L_1, L_2, L_3 \) must be unstable.

For \( L_4 \), however, we see that the linearized equation of motion are

\[ \ddot{x} - 2\dot{y} = \frac{3}{4} x + \frac{3\sqrt{3}}{4} (1 - 2\mu) y \]
\[ \ddot{y} + 2\dot{x} = \frac{9}{4} y + \frac{3\sqrt{3}}{4} (1 - 2\mu) x \]

And therefore the trial solution

\[ x = A e^{\omega t} \]
\[ y = B e^{\omega t} \]

gives:

\[ \omega^4 + \omega^2 + \frac{27}{4} \mu (1 - \mu) = 0 \]

Because

\[ \omega_1^2 \omega_2^2 = \frac{27}{4} \mu (1 - \mu) > 0 \]
The possible real roots have the same sign, furthermore since:

$$\omega_1^2 + \omega_2^2 = -1 < 0$$

Both roots \( \omega_2 \) must be negative. Since we require that

$$\omega^4 + \omega^2 + \frac{27}{4} \mu(1-\mu) = 0 \text{ has real roots and then }$$

\( \omega \) is imaginary,

we:

$$27\mu^2 - 27\mu + 1 > 0$$

This happens when

$$\mu < \frac{1}{2} - \sqrt{\frac{2}{\sqrt{108}} \approx .0385}$$

So \( L_4 \) and \( L_5 \) have stable orbits when \( \mu < .0385 \).
Appendix E: DF of Three-Body Problem in Rotating Inertial Frame

Equations of Motion in Rotating Inertial Frame

\[ \begin{align*}
\frac{d^2\mathbf{r}}{dt^2} &= -\nabla V \\
\frac{d^2\mathbf{v}}{dt^2} &= -\nabla V \\
\frac{d^2\mathbf{a}}{dt^2} &= -\nabla V
\end{align*} \]
\[ \dot{p} = 2q + \bar{z} - (1 - \mu)(\bar{z} + \mu) \left[ (\bar{z} + \mu)^2 + n^2 \right]^{-\frac{3}{2}} - \mu (\bar{z} - 1 + \mu) \left[ (\bar{z} - 1 + \mu)^2 + n^2 \right]^{-\frac{3}{2}} \]

\[ \frac{\partial \dot{p}}{\partial \bar{z}} = 1 - \bar{z} \left[ (\bar{z} + \mu)^2 + n^2 \right]^{-\frac{3}{2}} - \mu (\bar{z} - 1 + \mu) \left[ (\bar{z} - 1 + \mu)^2 + n^2 \right]^{-\frac{3}{2}} \]

\[ \frac{\partial \dot{\bar{z}}}{\partial \bar{z}} = - (1 - \mu)(\bar{z} + \mu) \left[ (\bar{z} + \mu)^2 + n^2 \right]^{-\frac{3}{2}} (2n) - \mu (\bar{z} - 1 + \mu) \left[ (\bar{z} - 1 + \mu)^2 + n^2 \right]^{-\frac{3}{2}} (2n) \]

\[ \dot{q} = -2p + n - n(1 - \mu) \left[ (\bar{z} + \mu)^2 + n^2 \right]^{-\frac{3}{2}} - \mu \left[ (\bar{z} - 1 + \mu)^2 + n^2 \right]^{-\frac{3}{2}} \]

\[ \frac{\partial \dot{q}}{\partial \bar{z}} = - n(1 - \mu) \left[ (\bar{z} + \mu)^2 + n^2 \right]^{-\frac{3}{2}} (2n) - \mu (1 - \mu) \left[ (\bar{z} - 1 + \mu)^2 + n^2 \right]^{-\frac{3}{2}} (2n) \]

\[ \frac{\partial \dot{q}}{\partial n} = 1 - \bar{z} \left[ (\bar{z} + \mu)^2 + n^2 \right]^{-\frac{3}{2}} + \mu (1 - \mu) \left[ (\bar{z} + \mu)^2 + n^2 \right]^{-\frac{3}{2}} (2n) + \mu (\bar{z} - 1 + \mu) \left[ (\bar{z} - 1 + \mu)^2 + n^2 \right]^{-\frac{3}{2}} (2n) \]