Density of the Rationals and Irrationals in $\mathbb{R}$

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Introduction

This note is intended to prove two facts which were listed on the first problem of the take-home exam, namely that if $a < b$ in $\mathbb{R}$, then there exists $x \in (a, b) \cap \mathbb{Q}$ and $y \in (a, b) - \mathbb{Q}$.

Archimedean Property of the Integers

First we prove the oft-assumed result that the sequence $\frac{1}{n} \to 0$ in the standard topology on $\mathbb{R}$. We need something called the *least upper bound property* of the real numbers.

Definition Let X be an ordered set. We say that $Y \subset X$ is bounded from above if there exists $x \in X$ such that $x \leq y$ for all $x \in X$, which we sometimes write as $Y \leq x$. We say that X has the least upper bound property if for any $Y \subset X$ which is bounded above, the collection $\{x \in X : Y \leq x\}$ of upper bounds for Y has a least element.

Theorem 1. *The real numbers have the least upper bound property.*

Proof. This theorem is usually taken for granted ("as an axiom") in most math classes. You can prove it, though. See the book "Classical Set Theory for Guided Self Study" by Derek Goldrei (which is delightful) for a thorough explanation. □

Theorem 2. *Let $Y \subset \mathbb{R}$ be bounded from above, and let $x \in \mathbb{R}$ an upper bound for Y, i.e. $Y \leq x$. Then TFAE*

1. *x is the least upper bound of Y.*

2. *For all $\epsilon > 0$, there exists $y \in Y$ in $(x - \epsilon, x]$.*

Proof. (1 implies 2): We prove by contrapositive. Suppose that there exists $\epsilon > 0$ such that $(x - \epsilon, x] \cap Y = \emptyset$. Then $x - \frac{\epsilon}{2}$ is also an upper bound for Y, and it is less than x. Thus x is not the least upper bound for Y.

(2 implies 1): We prove by contradiction. Suppose that $z < x$ is an upper bound for Y. Then $(x - (z - x), x]$ contains no points of Y, so that 2 does not hold. □

Theorem 3. *(Archimedean Property) Let $\mathbb{N} \subset \mathbb{R}$ be the positive integers. Then for any $x \in R$, there exists $n \in \mathbb{N}$ such that $x < n$.*

Proof. We contradict. Suppose that there is some $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. Then $\mathbb{N}$ is a bounded from above, and hence has a least upper bound $y \in \mathbb{R}$. Take an element m of $\mathbb{N}$ such that $m \in (y - 1/2, y]$ by the above theorem. Then $y - 1/2 < m \leq y$. Adding 1 to this inequality we obtain $y + 1/2 < m + 1$. As $m + 1$ is an integer, we see that y is not an upper bound for $\mathbb{N}$. $\square$

Corollary 1. *Let $a_n = \frac{1}{n}$. Then $a_n \to 0$ in the standard top on $\mathbb{R}$.*

Proof. Let $U = (-\epsilon, \epsilon)$ be a basis element containing 0. We show that U contains all but finitely many terms of the sequence. Take an integer $N > \frac{1}{\epsilon}$. Then for all $n > N$, $0 < a_n = \frac{1}{n} < \frac{1}{N} < \epsilon$, so that $a_n \in U$ for all $n > N$. $\square$

Density of the Rationals

Theorem 4. *Let $a < b$ in $\mathbb{R}$. Then there is a rational number in (a, b).*

Proof. Let $\epsilon = b - a$. Pick $N \in \mathbb{N}$ such that $N > \frac{1}{b-a}$, whence $\frac{1}{N} < b - a$. Let $A = \{\frac{m}{N} : m \in \mathbb{N}\}$, a subset of $\mathbb{Q}$. Claim: $A \cap (a, b) \neq \emptyset$. Assume otherwise, so that we can take m_1 the greatest integer such that $\frac{m_1}{N} < a$. Then $\frac{m_1+1}{N} > b$. But this implies that $b - a < \frac{m_1+1}{N} - \frac{m_1}{N} = \frac{1}{N} < b - a$, a contradiction. Thus $(a, b) \cap \mathbb{Q} \neq \emptyset$. $\square$

Density of the Irrationals

In order to prove that the irrationals are dense, we first need to show that there *exist*. That there are irrational numbers, and that specific numbers such as $\sqrt{2}$ and π are irrational, is a favorite fact of mathematicians.

Lemma 1. *If $0 < x < 1$ is a real number, the sequence $a_n = x^n$ converges to 0.*

Proof. We prove that the sequence $b_n = \frac{1}{x^n}$ diverges to ∞. Taking reciprocals then proves the desired statement. As $x < 1$, we have $\frac{1}{x} = 1 + y$, where $y > 0$. Note that $(1 + y)^n = 1 + ny + \binom{n}{2}y^2 + \ldots + y^n$. Truncating this, we have $(1 + y)^n \geq 1 + ny$. Then for any $z \in \mathbb{R}$, we can find $N > \frac{z-1}{y}$. Then $y > \frac{z-1}{N}$. Then $n > N$ implies that

$$(1 + y)^n \geq 1 + ny > 1 + \frac{n(z-1)}{N} > 1 + z - 1 = z$$

where in the penultimate equation we used $\frac{n}{N} > 1$. So $(1 + y)^n$ diverges to ∞, so that $a_n \to 0$. $\square$

Theorem 5. *The number $\sqrt{2}$ is not rational.*

Proof. Note that $1 < \sqrt{2} < 2$, as $1 < 2 < 4$ and the square-root function increases. Thus $0 < \sqrt{2} - 1 < 1$.

Suppose that $\frac{p}{q} = \sqrt{2}$, where p and q are positive integers without any common prime factors. Define $a_n = (\sqrt{2} - 1)^n = (\frac{p}{q} - \frac{q}{q})^n = (\frac{p-q}{q})^n$. Then by the previous lemma, a_n converges to 0.

Now here's the cool part: Every term $a_n = \frac{x_n p + y_n q}{q}$ for some choice of integers x_n, y_n. Prove this by induction: if $a_n = \frac{x_n p + y_n q}{q}$, then

$$\begin{aligned} a_{n+1} &= \left(\frac{p-q}{q}\right)\left(\frac{x_n p + y_n q}{q}\right) \\ &= \frac{x_n p^2 + y_n pq - x_n pq - y_n q^2}{q^2} \\ &= 2x_n - y_n + \frac{y_n p - x_n p}{q} \\ &= \frac{(2x_n - y_n)q + (y_n - x_n)p}{q} \end{aligned}$$

So a_{n+1} is also a rational number of this form. Now the contradiction: if $0 < a_n = \frac{x_n p + y_n q}{q}$, then $a_n \geq \frac{1}{q}$, as the numerator is an integer and hence is at least 1. But this implies that a_n does *not* converge to 0. This contradiction forces $\sqrt{2}$ to be irrational.

□

Corollary 2. *If $a < b$ in $\mathbb{R}$, then there exists $x \in (a, b) - \mathbb{Q}$.*

Proof. Take a non-zero rational number $q \in (\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}})$. If $a < 0 < b$, find a rational number in $(\frac{a}{\sqrt{2}}, 0)$. Then $\sqrt{2}q \in (a, b)$. If $\sqrt{2}q = q'$ is rational, then $\sqrt{2} = \frac{q'}{q}$, also a rational number. Thus $\sqrt{2}q$ is irrational, and every interval contains an irrational number. □