MATH 54 - EXAM 1: IN-CLASS EXAM

Name (Print): ____
Last First

On this exam I will work individually, neither giving nor receiving help, guided by the Dartmouth Academic Honor Principle.

Signature:____

Problem 1 (4 pts each - 16 total)

Define each of these. Any topologies which are defined in terms of explicit bases may be described that way.

(a) A topology on a set X

A collection τ of subsets of X such that:
(a) $X, \emptyset \in \tau$**.**
(b) If $\{U_i\}_{i \in I} \subset \tau$**, then** $\cup_i U_i \in \tau$**.**
(c) If $\{U_k\}_{k=1}^n$ **is a finite subcollection of** τ**, then** $\cap_{k=1}^n U_k \in \tau$**.**

(b) Given a basis $\mathcal{B}$ of subsets of X, the topology generated by $\mathcal{B}$.

A set $U \subset X$ **is open if for every** $x \in U$ **there exists** $B \in \mathcal{B}$ **such that** $x \in B \subset U$**. Equivalently,** U **is open if it is a union of some of the elements in** $\mathcal{B}$**.**

(c) A closed subset C of a topological space X.

C **is closed if** $X - C$ **is open.**

(d) A continuous function $f : X \to Y$, where X and Y are topological spaces.
f is continuous if for every $U \subset Y$, the preimage $f^{-1}(U)$ is open in X.

Problem 2 (2 pts each - 10 total): For each of the following, circle True or False. Make sure there is no ambiguity as to which you have selected. You do NOT have to provide justification for your answer.

(1) Suppose X and Y are topological spaces. A function $f : X \to Y$ is continuous, provided that $f^{-1}(C)$ is closed for any closed subset $C \subset Y$.

TRUE FALSE .

(2) Any basis of subsets of a set X is also a topology on X.

TRUE **FALSE** .

(3) If $\{X_i\}_{i\in I}$ is a collection of spaces, then the box topology on $\prod_{i\in I} X_i$ contains the product topology on $\prod_{i\in I} X_i$.

TRUE FALSE .

(4) The discrete topology on a set X is finer than any other topology on X.

TRUE FALSE .

(5) Suppose $\{\tau_i\}_{i\in I}$ is a collection of topologies on the same set X. Then $\cup_{i\in I}\tau_i$ is the smallest topology on X which contains all the topologies τ_i

TRUE **FALSE** .

Problem 3 (2 pt each - 8 total) Provide an example of each of the following. You do not need to justify your answers.

(a) A topological space.

The reals with the standard topology.

(b) A continuous function between two topological spaces.

The projection map $\pi_1 : X \times Y \to X$**, where** X **and** Y **are spaces. Any constant map** $f : \mathbb{R} \to \mathbb{R}$**.**

(c) A subset of $\mathbb{R}$ which is neither open nor closed in the standard topology.

$[0, 1)$.

(d) Two incomparable topologies on a set X.

The topologies from $\mathbb{R}_K$ and $\mathbb{R}_\ell$ are incomparable topologies on $\mathbb{R}$.

Problem 4 (2 pts each - 4 total) Let X be a topological space.

(a) Prove that if X has the discrete topology, then every map $f : X \to Y$, where Y is another topological space, is continuous

Let $U \subset Y$ be open. Then $f^{-1}(U) \subset X$. As every subset of X is open in the discrete topology, $f^{-1}(U)$ is open. Thus f is continuous.

(b) Prove that if X has the concrete/indiscrete topology, then every map $g : Y \to X$, where Y is another topological space, is continuous.

Let $U \subset X$ be open. Then $U = X$ or $U = \emptyset$, by definition of the indiscrete topology. If $U = X$, then $g^{-1}(U) = g^{-1}(X) = Y$. If $U = \emptyset$, then $g^{-1}(U) = g^{-1}(\emptyset) = \emptyset$. In both of these cases $g^{-1}(U)$ is open. Thus g is continuous.

Problem 5 (4 pts each - 8 total)
Let $X = \{a, b, c\}$ and let $\tau_1 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_2 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$.

(a) What is the largest topology on X contained in both τ_1 and τ_2?

The intersection $\tau_1 \cap \tau_2 = \{X, \emptyset, \{a\}\}$ is the smallest topology contained in both.

(b) What is the smallest topology on X containing both τ_1 and τ_2?

Such a topology τ must contain $\tau_1 \cup \tau_2 = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}\}$. As a topology is contained under unions, we must have $\{b\} \cup \{c\} = \{b, c\} \in \tau$. This compels $\tau = \mathcal{P}(X)$, the discrete topology. (Note: you did not need to justify your answer to receive credit on this problem).

Problem 6 (4 pts)
Let $\mathbb{R}$ denote the real line equipped with its standard topology and let $\mathbb{R}'$ denote the real line equipped with the cofinite (finite complement) topology. Prove or disprove: the identity function $i : \mathbb{R}' \to \mathbb{R}$ defined by the rule $i(x) = x$ is continuous.

Let $U = (0, 1) \subset \mathbb{R}$. Then U is open in the standard topology. The preimage under the identity map is the same set, i.e. $i^{-1}(0, 1) = (0, 1)$. However, the complement of $(0, 1)$ is $(-\infty, 0] \cup [1, \infty)$. As this set is neither finite nor all of $\mathbb{R}$, $(0, 1)$ is not open in $\mathbb{R}'$. Thus i is not continuous.