# MATH 54: FIRST TAKE-HOME EXAM - DUE FRIDAY, JULY 20TH

**Name (Print):** ______________________________________________
Last First

**Instructions**: You are neither to give nor receive any assistance on this exam. If you have questions about something please email me. You can use: notes from class, the lecture notes available on the course website, homework solutions (yours or those from the course website), and Munkres's book. You cannot use: the Internet (besides the aforementioned resources), your classmates, other books, notes, or papers, or any other resource, printed, living, or electronic.

Note: if you cannot fit your solution on the paper provided, please attach a neatly handwritten solution to the paper and indicate on the page of the problem where to find your solution. Label any attached pages so that solutions to problems can be easily found.

On this exam I will work individually, neither giving nor receiving help, guided by the Dartmouth Academic Honor Principle.

**Signature:**___________________________________

**Problem 1** (3 pts each - 15 pts total): Consider the following three topologies on $\mathbb{R}$: $\tau_1$ is the countable complement topology, i.e. $\tau_1 = \{U \subset \mathbb{R} : \mathbb{R} - U \text{is countable or all of } \mathbb{R}\}$, $\tau_2$ is the lower limit topology, and $\tau_3$ is the $K$-topology. For each of the following subsets of $\mathbb{R}$, determine (with proof) whether or not it is open in each of the three topologies. If a topology is defined using a basis, you may refer to that basis.

Two facts about $\mathbb{R}$:

(1) $\mathbb{R}$ is uncountable.

(2) If $x < y$ in $\mathbb{R}$ then there exists $p \in \mathbb{Q} \cap (x, y)$ and $r \in (x, y) - \mathbb{Q}$.

(a) $\cup_{n=1}^{\infty}(\frac{1}{n+1}, \frac{1}{n}) \cup (-1, 0]$.

**This is the set $(-1, 1) - K$. Its complement in $\mathbb{R}$ contains $(-\infty, -1]$, hence its complement is uncountable. Thus it is not open in $\tau_1$. No half-open interval $[0, \epsilon)$, where $\epsilon > 0$, can exclude all points of $K$. Hence $(-1, 1) - K$ is not open in the lower limit topology $\tau_2$. As $(-1, 1) - K$ is a basis element for the $K$-topology, it is open in $\tau_3$.**

(b) The set $\mathbb{R} - \mathbb{Q}$ of all irrational numbers.

**The complement of $\mathbb{R} - \mathbb{Q}$ in $\mathbb{R}$ is the set $\mathbb{Q}$. This set is countable, hence $\mathbb{R} - \mathbb{Q}$ is open in $\tau_1$. Each basis element of $\tau_2$ and $\tau_3$ contains $\mathbb{Q}$. Thus no union of such elements can equal $\mathbb{R} - \mathbb{Q}$. So $\mathbb{R} - \mathbb{Q}$ is not in $\tau_2$ or $\tau_3$.**

(c) The set $(0, 1]$.

**This set is not open in any of the topologies. The complement contains $(-\infty, 0]$ which is uncountable. Thus $(0, 1] \notin \tau_1$. Any basis element in $\tau_2$ or $\tau_3$ which contains $0$ also contains a positive number. Thus $(0, 1]$ is not open in either of these topologies.**

(d) The set $(-\infty, 1/2) \cup [2/3, \infty)$.

**The set is not in $\tau_1$ as the complement $[1/2, 2/3)$ is uncountable. The set $(-\infty, 1/2)$ is open in $\mathbb{R}_\ell$, as is $[2/3, 1) \cup (2/3, \infty)$. Thus the set is in $\tau_2$. Any basis element of the $K$-topology which contains $2/3$ also contains a number from $(1/2, 2/3)$. Thus the set is not open in $\tau_3$.**

(e) The set $(-\infty, \pi) \cup (\pi, \infty)$.

**This set is open in all of the topologies. Its complement is $\{\pi\}$, a finite and hence countable set, so it is included in $\tau_1$. It is an open set in the standard topology on $\mathbb{R}$, so it must be an element of $\tau_2$ and $\tau_3$, as each of these topologies contains the standard topology.**

**Problem 2** (2 pts each - 10 total): In which of the following topologies on $\mathbb{R}$ does $a_n = \frac{(-1)^n}{n}$ converge to 0? Justify your answers.

(a) The $K$-topology.

**Does not converge: $a_{2n} \notin (-1,1) - K$ for all $n$. Thus there is no integer $N$ such that $m > N$ implies $a_m \in (-1,1) - K$.**

(b) The lower limit topology.

**Does not converge: The set $[0,1)$ is a nbhd of $0$ in the lower limit topology. $a_{2n+1} \notin [0,1)$ for all $n$. Thus no convergence for the same reason.**

(c) The discrete topology.

**Does not converge: The set $\{0\}$ is a nbhd of $0$ in the discrete topology. $a_n \notin \{0\}$ for all $n$. Hence no $N$ exists such that $n > N$ implies $a_n \in \{0\}$.**

(d) The ray topology generated by the basis $\{(-\infty, a) : a \in \mathbb{R}\}$.

**Converges: Let $(-\infty, a)$ be a basic set containing $0$. Then $a > 0$. Pick $N > 1/a$. If $n > N$, then $|a_n| = |1/n| < |1/N| < a$, so that $a_n \in (-\infty, a)$.**

(e) The cofinite topology. **Converges: let $U$ be a nbhd of $0$ in cofinite topology. Let $\mathbb{R} - U = \{x_1, \ldots, x_k\}$. Let $N$ be the largest integer such that $a_N \in \{x_1, \ldots, x_k\}$ (exists as all the terms of the sequence are distinct). If $n > N$, then $a_n \in U$. Thus $a_n \to 0$.**

**Problem 3** (5 pts each - 15 pts total)

(1) Let $X$ and $Y$ be topological spaces with $X \cap Y = \emptyset$. Let $\tau$ be the collection of all subsets of $X \cup Y$ of the form $U \cup V$, where $U$ is an open subset of $X$ and $V$ is an open subset of $Y$. Prove that $\tau$ is a topology on $X \cup Y$.

**We see that $X \cup Y$ is the union of $X$ an open set in $X$ and $Y$ an open set in $Y$. Similarly $\emptyset = \emptyset \cup \emptyset$.**

**Suppose that $\{U_i \cup V_i\}_{i \in I}$ is a family of open sets in $X \cup Y$. Then $\cup_{i\in I}(U_i \cup V_i) = (\cup_{i \in I} U_i) \cup (\cup_{i\in I} V_i)$ is an open set in $X \cup Y$ as well, as $X$ and $Y$ are topologies and hence closed under unions. Thus the proposed topology on $X \cup Y$ is closed under arbitrary unions.**

**Suppose that $\{U_k \cup V_k\}_{k=1}^n$ is a finite collection. Then $\cap_{k=1}^n U_k \cup V_k = \cap_{k=1}^n U_k \cup \cap_{k=1}^n V_k$, for all the intersections $U_i \cap V_j$ are empty. Thus the proposed topology on $X \cup Y$ is closed under finite intersections. So the collection is a topology on $X \cup Y$.**

(2) Prove that the inclusion maps $i_X : X \to X \cup Y$ and $i_Y : Y \to X \cup Y$ are both continuous.

**Let $W = U \cup V$ be an open set in $X \cup Y$, so that $U$ is an open set of $X$ and $V$ is an open set of $Y$. Then $i_X^{-1}(W) = U$ and $i_Y^{-1}(W) = V$ are both open. Thus the inclusion maps are continuous.**

(3) Suppose that $f : X \to Z$ and $g : Y \to Z$ are continuous functions into some space $Z$. Prove that the map $f \cup g : X \cup Y \to Z$ given by $(f \cup g)(x) = f(x)$ for $x \in X$ and $(f \cup g)(y) = g(y)$ for $y \in Y$ are continuous. (Hint: We proved a lemma about making continuous functions out of smaller continuous functions).

**The domains of $f$ is $X = X \cup Y - Y$, a closed set as $Y = \emptyset \cup Y$ is open. Similarly the domain of $g$ is closed. The functions vacuously agree on the intersection $X \cap Y = \emptyset$. Thus the pasting lemma tells us that $f \cup g$ is continuous.**

**Problem 4** (10 pts): Suppose that $X$ is a space and $A \subset X$. Equip $A$ with the subspace topology. Suppose that $f : A \to Y$ is a continuous function into a Hausdorff space $Y$. Show that if $g : \overline{A} \to Y$ and $h : \overline{A} \to Y$ are continuous functions such that $g|_A = f = h|_A$, then $g = h$ on $\overline{A}$. (Here $\overline{A}$ also has the subspace topology).

**Suppose that the statement is false, so that there exists $x \in \overline{A}$ such that $g(x) \neq h(x)$. As $Y$ is Hausdorff, we can produce disjoint nbhds $U$ and $V$ of $g(x)$ and $h(x)$ respectively. Thus $x \in g^{-1}(U)$ and $x \in h^{-1}(V)$, both nbhds in $\overline{A}$. Then $x \in g^{-1}(U) \cap h^{-1}(V)$, which as a finite intersection of open sets is another nbhd of $x$ in $\overline{A}$. By the definition of the subspace topology, there must be an open set $P \subset X$ such that $g^{-1}(U) \cap h^{-1}(V) = \overline{A} \cap P$. Thus $P$ is a nbhd of $x$ in $X$. As $x \in \overline{A}$, there is a point of $a \in P$. Thus $a \in P \cap A \subset P \cap \overline{A} = g^{-1}(U) \cap h^{-1}(V)$. Then $g(a) \in U$ and $h(a) \in V$. But $g|_A = h|_A$, so $g(a) = h(a)$, which yields $U \cap V \neq \emptyset$, a contradiction. Thus the statement must be true.**