MATH 54 - HW SOLUTIONS THE FIRST

DAN CRYTSER

Prob 1.1

Let V be a set, $\{U_i\}_{i\in I}$ a collection of sets.

(a) We prove the distributive law for union. First we show that $V \cup (\cap_{i\in I} U_i) \subset \cap_{i\in I} V \cup U_i$. Suppose that $x \in V \cup (\cap_{i\in I} U_i$. If $x \in V$, then $x \in V \cup U_i$ for all i, and so $x \in \cap_{i\in I} V \cup U_i$. So $V \subset \cap_{i\in I} V \cup U_i$. If $x \in \cap_{i\in I} U_i$, then for all i, $x \in U_i \subset V \cup U_i$. Thus $x \in \cap_{i\in I} V \cup U_i$. So $\cap_{i\in I} U_i \subset \cap_{i\in I} V \cup U_i$. So the union $V \cup \cap_{i\in I} U_i \subset \cap_{i\in I} V \cup U_i$.

We prove the reverse inclusion now: Now suppose that $x \in \cap_{i\in I} V \cup U_i$. For each i, x is in at least one of V or U_i. If there is no i such that $x \notin U_i$, then $x \in \cap_{i\in I} U_i \subset V \cup \cap_{i\in I} U_i$. Otherwise, there is some i such that $x \notin U_i$, and as $x \in V \cup U_i$, we must have $x \in V \subset V \cup \cap_{i\in I} U_i$. Thus $\cap_{i\in I} V \cup U_i \subset V \cup \cap_{i\in I} U_i$.

Since we have proven both inclusions, we see that $\cap_{i\in I} V \cup U_i = V \cup \cap_{i\in I} U_i$.

(b) Now we prove the distributive law for intersection, again with the same notation as before. First we show that $V \cap \cup_{i\in I} U_i \subset \cup_{i\in I} V \cap U_i$. Suppose that $x \in V \cap \cup_{i\in I} U_i$. Then in particular $x \in \cup_{i\in I} U_i$, so that there exists $j \in I$ such that $x \in U_j$. But $x \in V$, so that $x \in V \cap U_j$. But $V \cap U_j \subset \cup_{i\in I} V \cap U_i$, so that $x \in \cup_{i\in I} V \cap U_i$. Thus $V \cap \cup_{i\in I} U_i \subset \cup_{i\in I} V \cap U_i$.

Now we prove the reverse inclusion. Suppose that $x \in \cup_{i\in I} V \cap U_i$. Then there exists i such that $x \in V \cap U_i$. In particular, $x \in U_i$, and so $x \in \cup_{i\in I} U_i$. As $x \in V$, we see that $x \in V \cap \cup_{i\in I} U_i$. So $x \in \cup_{i\in I} V \cap U_i \subset V \cap \cup_{i\in I} U_i$.

(c) We prove the deMorgan law $V \setminus \cup_{i\in I} U_i = \cap_{i\in I} V \setminus U_i$. First we show that $V \setminus \cup_{i\in I} U_i \subset \cap_{i\in I} V \setminus U_i$. Let $x \in V \setminus \cup_{i\in I} U_i$. Then $x \in V$ and for all $i \in I$, we have $x \notin U_i$, or else $x \in \cup_{i\in I} U_i$. Thus $x \in V \setminus U_i$ for all $i \in I$. So $x \in \cap_{i\in I} V \setminus U_i$. Thus $V \setminus \cup_{i\in I} U_i \subset \cap_{i\in I} V \setminus U_i$

Now suppose that $x \in \cap_{i\in I} V \setminus U_i$. Then $x \in V$. Also, for each $i \in I$, we have $x \notin U_i$, so that $x \notin \cup_{i\in I} U_i$. So $x \in V \setminus \cup_{i\in I} U_i$. Thus $\cap_{i\in I} V \setminus U_i \subset V \setminus \cup_{i\in I} U_i$.

As we have proven both inclusions, we have that $\cap_{i\in I} V \setminus U_i = V \setminus \cup_{i\in I} U_i$.

(d) We prove the deMorgan law $V \setminus \cap_{i\in I} U_i = \cup_{i\in I} V \setminus U_i$. First we prove that $V \setminus \cap_{i\in I} U_i \subset \cup_{i\in I} V \setminus U_i$. Suppose that $x \in V \setminus \cap_{i\in I} U_i$. Then $x \notin \cap_{i\in I} U_i$, so that there is some $j \in I$ such that $x \notin U_j$. As $x \in V$, we have $x \in V \setminus U_j$, so that $x \in \cup_{i\in I} V \setminus U_i$. Thus $V \setminus \cap_{i\in I} U_i \subset \cup_{i\in I} V \setminus U_i$.

Now we prove the reverse inclusion. Suppose that $x \in \cup_{i\in I} V \setminus U_i$. Then there exists some $j \in I$ such that $x \in V \setminus U_j$. So $x \in V$ and $x \notin U_j$. Then $x \notin \cap_{i\in I} U_i$, and we have $x \in V \setminus \cap_{i\in I} U_i$. So we have $\cup_{i\in I} V \setminus U_i \subset V \setminus \cap_{i\in I} U_i$. We proved both inclusions and so we have $\cup_{i\in I} V \setminus U_i = V \setminus \cap_{i\in I} U_i$.

Prob 1.3

(a) Write the contrapositive and converse of the following statement: "If $x < 0$, then $x^2 - x > 0$," and determine which if any of the three statements are true.

The statement is true, as if $x < 0$, $x^2 > 0$, and so $x^2 - x > 0$.

The contrapositive is: If $x^2 - x \leq 0$, then $x \geq 0$. This is true, because the contrapositive of a statement is equivalent to the statement.

The converse is: if $x^2 - x > 0$, then $x < 0$. This is false, as $2^2 - 2 = 2 > 0$.

(b) We do the same for "If $x > 0$, then $x^2 - x > 0$."

The statement is false, as $(1/2)^2 - 1/2 < 0$.

The contrapositive is: If $x^2 - x \leq 0$, then $x \leq 0$. This is false, as the contrapositive of a statement is equivalent to the statement.

The converse is: If $x^2 - x > 0$, then $x > 0$. This is false, as $(-1)^2 - (-1) > 0$, and yet $-1 < 0$.

Prob 2.2

Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i \in \{0, 1\}$.

(a) We show that $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$. Suppose that $x \in f^{-1}(B_0)$. Then $f(x) \in B_0$, hence $f(x) \in B_1$. Thus $x \in f^{-1}(B_1)$, and we have $f^{-1}(B_0) \subset f^{-1}(B_1)$.

(b) We prove that $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$. First we show the inclusion $f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_0) \cup f^{-1}(B_1)$. Let $x \in f^{-1}(B_0 \cup B_1)$. Then $f(x) \in B_0 \cup B_1$. If $f(x) \in B_0$, then $x \in f^{-1}(B_0)$, and if $f(x) \in B_1$, then $x \in f^{-1}(B_1)$. So $x \in f^{-1}(B_0) \cup f^{-1}(B_1)$. Thus $f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_0) \cup f^{-1}(B_1)$.

Now we show the reverse inclusion. If $x \in f^{-1}(B_0) \cup f^{-1}(B_1)$, then $x \in f^{-1}(B_0)$ or $x \in f^{-1}(B_1)$. In the first case $f(x) \in B_0$ and in the second case $f(x) \in B_1$. In either case, $f(x) \in B_0 \cup B_1$, so that $x \in f^{-1}(B_0 \cup B_1)$. Thus $f^{-1}(B_0) \cup f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1)$. Thus we have shown both inclusions and we get $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.

(c) We prove that $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$. [Now that we have done a few of these we will skip some of the language in the previous examples]. Suppose $x \in f^{-1}(B_0 \cap B_1)$. Then $f(x) \in B_0 \cap B_1$. So $f(x) \in B_0$, so that $x \in f^{-1}(B_0)$, and $f(x) \in B_1$, so that $x \in f^{-1}(B_1)$. So $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$.

Now suppose that $x \in f^{-1}(B_0) \cap f^{-1}(B_1)$. Then $x \in f^{-1}(B_0)$, so that $f(x) \in B_0$. Similarly, $f(x) \in B_1$. So $f(x) \in B_0 \cap B_1$, and $x \in f^{-1}(B_0 \cap B_1)$. Thus both inclusions are proven, and we have the equation $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

(d) We prove the equation $f^{-1}(B_0 \setminus B_1) = f^{-1}(B_0) \setminus f^{-1}(B_1)$. Suppose that $x \in f^{-1}(B_0 \setminus B_1)$. Then $f(x) \in B_0 \setminus B_1$. Then $f(x) \in B_0$, so that $x \in f^{-1}(B_0)$. Also, $f(x) \notin B_1$, so that $x \notin f^{-1}(B_1)$. Thus $x \in f^{-1}(B_0) \setminus f^{-1}(B_1)$.

Now suppose that $x \in f^{-1}(B_0) \setminus f^{-1}(B_1)$. Then $f(x) \in B_0$ and $f(x) \notin B_1$. Thus $x \in f^{-1}(B_0 \setminus B_1)$. Thus both inclusions are proven and the equation holds.

(e) We prove that $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$. Suppose that $y \in f(A_0)$. Then $y = f(x)$ for some $x \in A_0$. Then $x \in A_1$ also, so that $y \in f(A_1)$. Thus $f(A_0) \subset f(A_1)$.

(f) We prove that $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$. Let $y \in f(A_0 \cup A_1)$, so that $y = f(x)$ for some $x \in A_0 \cup A_1$. If $x \in A_0$, then $f(x) \in f(A_0)$, and if $x \in A_1$ then $f(x) \in f(A_1)$, so that $y = f(x) \in f(A_0) \cup f(A_1)$. Now suppose that $y \in f(A_0) \cup f(A_1)$.

If $y \in f(A_0)$, then $y = f(x)$ for some $x \in A_0$, and if $y \in f(A_1)$, then $y = f(x)$ for some $x \in A_1$. In either case, $y \in f(A_0) \cup f(A_1)$. Thus we have shown both inclusions and the equation holds.

(g) We prove that $f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1)$, and we prove the reverse inclusion in the case where f is injective. Suppose that $y \in f(A_0 \cap A_1)$, so that there exists $x \in A_0 \cap A_1 \subset A_0$ such that $f(x) = y$. Then $x \in A_0$, so that $f(x) \in f(A_0)$, and similarly $f(x) \in f(A_1)$. Thus $y \in f(A_0) \cap f(A_1)$. So the inclusion holds.
Now assume that f is injective. Suppose that $y \in f(A_0) \cap f(A_1)$. Then there exists $x_0 \in A_0$ such that $y = f(x_0)$ and there exists $x_1 \in A_1$ such that $f(x_1) = y$. Then $f(x_0) = f(x_1)$. But f is injective, so that $x_0 = x_1$. This implies that $x_0 \in A_0 \cap A_1$, as $x_1 \in A_1$. Thus $y = f(x_0)$ with $x_0 \in A_0 \cap A_1$, so that $y \in f(A_0 \cap A_1)$. Thus the inclusion $f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1)$ holds when f is injective. (Note: a counterexample for the non-injective case is given by $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ and $A_0 = [-1, 0]$, $A_1 = [0, 1]$).

(h) We prove that $f(A_0 \setminus A_1) \supset f(A_0) \setminus f(A_1)$ and we prove the reverse inclusion when f is injective. Suppose that $y \in f(A_0) \setminus f(A_1)$. Then $y = f(x)$ for some $x \in A_0$, and $y \neq f(x')$ for all $x' \in A_1$. Thus $x \notin A_1$, and we have $x \in A_0 \setminus A_1$. Thus $y = f(x)$ for $x \in A_0 \setminus A_1$, and $y \in f(A_0 \setminus A_1)$.
Now we assume that f is injective. Suppose that $y \in f(A_0 \setminus A_1)$. Then $y = f(x)$ for some $x \in A_0 \setminus A_1$. This gives $y \in f(A_0)$. If $y' = f(x')$ for some $x' \in A_1$, then we have $y \neq y'$, for otherwise $f(x) = f(x')$ and we conclude that $x = x'$, contradicting $x \notin A_1$. Thus $y \in f(A_0) \setminus f(A_1)$. So the reverse inclusion holds when f is injective

Problem 4.4

(a) We prove by induction that if $n \in \mathbb{N}$ is a positive integer and $C \subset \{1, \ldots, n\}$ is nonempty, then C has a largest element.
Let A be the set of all integers n for which the statement is true. Then $1 \in A$, for the set $\{1\}$ has the largest element 1. Now suppose that $n \in A$. We prove that $n + 1 \in A$. Suppose that $C \subset \{1, \ldots, n + 1\}$ is non-empty. If $n + 1 \in C$, then $n + 1$ is the largest element of C and the statement holds. If $n + 1 \notin C$, then $C \subset \{1, \ldots, n\}$, and the inductive hypothesis tells us that C has a largest element. Thus A is inductive, and as A contains 1, we have that $A = \mathbb{N}$. So if $C \subset \{1, \ldots, n\}$ for some positive integer n, then C has a largest element.

(b) We cannot conclude that every nonempty subset of $\mathbb{N}$ has a largest element because not every subset of $\mathbb{N}$ is included in $\{1, \ldots, n\}$ for some n. For example $\mathbb{N} \not\subset \{1, \ldots, n\}$ for all n.

Prob 6.2

We show that if B is not finite and $B \subset A$, then A is not finite. Corollary 6.6 says that all subsets of a finite set are finite. Taking the contrapositive of this says that if there exists an infinite subset of A, then A is infinite.

Euclidean Ball Problem

Let $x \in \mathbb{R}^n$ for some n, and suppose $\epsilon > 0$. We prove that the open ball $B(x, \epsilon)$ is open in $\mathbb{R}^n$. Let $y \in B(x, \epsilon)$. Then $d = ||x - y|| < \epsilon$. Let $\delta = \epsilon - d$. Claim: $B(y, \delta) \subset B(x, \epsilon)$. For suppose that $z \in B(y, \delta)$. Then

$$||x - z|| = ||x - y + y - z|| \leq ||x - y|| + ||y - z|| < d + \epsilon - d = \epsilon.$$

Thus $z \in B(x, \epsilon)$, and we have $B(y, \delta) \subset B(x, \epsilon)$. Thus about an arbitrary point of the set $B(x, \epsilon)$ we have inscribed an open ball contained within $B(x, \epsilon)$, and so $B(x, \epsilon)$ is open.