MATH 54 - HOMEWORK SOLUTIONS THE SECOND

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Problem 13.1

Let X be a topological space and let $A \subset X$. Suppose that for each $x \in A$ there exists an open subset U such that $x \in U \subset A$. Show that A is an open set.

For each x, let U_x be an open set in X such that $x \in U_x \subset A$. Then $A = \cup_{x \in A} U_x$, as $U_x \subset A$ for all x and $y \in A$ belongs to $U_y \subset \cup_{x \in A} U_x$. As A is the union of a collection $\{U_x\}_{x \in A}$ of open sets, it must also be open.

Problem 13.3

Let X be a set, let τ_c be the collection of all subsets $U \subset X$ such that $X - U$ is either countable or all of X. Check that this defines a topology on X.

First we check that $X, \emptyset \in \tau_c$. As the complement of X in X is the emptyset, which is finite and hence countable, we have $X \in \tau_c$. The complement of $\emptyset$ is all of X, which is in τ_c.

Now we check that the union of $\{U_i\}_{i \in I} \subset \tau_c$ is contained in τ_c. Let $U = \cup_{i \in I} U_i$. Then $X - U = \cap_{i \in I} X - U_i$ by DeMorgan's laws. If all of the complements $X - U_i$ equal X, then so does their intersection. If one of the complements $X - U_i$ is countable, then so is $X - U \subset X - U_i$, as any subset of a countable set is countable. Thus $X - U$ is either countable or all of X, and so U is in τ_c.

Finally we check that the finite intersection of elements from τ_c is contained in τ_c. Let $U_1, \ldots, U_n \in \tau_c$, and consider $U = \cap_{i=1}^n U_i$. Then $X - U = \cup_{i=1}^n X - U_i$ by DeMorgan's laws. If any of the sets $X - U_i$ is the total set X, then so is the union $X - U$, as $X - U_i \subset X - U$. If all of the sets $X - U_i$ are countable, then so is the finite union of countable sets $X - U$, by theorem 7.5. Thus $X - U$ is either all of X or countable, and so $U \in \tau_c$.

The collection $\tau_\infty = \{U | X - U \text{ is infinite or empty or all of } X\}$ is not a topology on an infinite set X. For each $x \in X$, the singleton $\{x\} \in \tau_\infty$ if X is infinite. Yet for any fixed $y \in X$ the union $\cup_{x \neq y} \{x\}$ is a union of sets in τ_∞ which is not contained in τ_∞, as its complement is the finite set $\{y\}$.

Problem 13.4

(a) If $\{\tau_{i \in I}\}$ is a family of topologies on X, show that $\cap \tau_{i \in I}$ is a topology on X. Is $\cup \tau_{i \in I}$ a topology on X?

We let $\tau = \cap_{i \in I} \tau_i$. Then $X, \emptyset \in \tau$, as $X, \emptyset \in \tau_i$ for all i (each τ_i is a topology).

Suppose that $\{U_j\}_{j \in J}$ is a collection of sets in τ. Then for each $i \in I$, $\{U_j\}_{j \in J} \subset \tau_i$, as the collection lies in the intersection. As τ_i is a topology, the union $\cup_{j \in J} U_j$ is an element of τ_i. Thus $\cup_{j \in J} U_j \in \tau_i$ for all $i \in I$, and $\cup_{j \in J} U_j \in \tau$.

Suppose that $U_1, \ldots, U_n$ is a finite collection of sets in τ. Then the intersection $\cap_{k=1}^n U_k$ is contained in each τ_i, as each τ_i contains the sets $\{U_k\}_{k=1}^n$ and all finite intersections of its sets. Thus τ is closed under finite intersections.

If we consider the two topologies on $X = \{a, b, c\}$ given by $\tau_1 = \{X, \emptyset, \{a\}\}$ and $\tau_2 =$

$\{X, \emptyset, \{b\}\}$, then their union is $\tau = \{X, \emptyset, \{a\}, \{b\}\}$. But τ does not contain the union $\{a, b\} = \{a\} \cup \{b\}$, so that τ is not a topology.

(b) Let $\{\tau_i\}_{i \in I}$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all of the τ_i, and a unique largest topology on X contained in all the τ_i.
We prove that there is a smallest topology containing all the τ_i. The collection $\mathcal{C}$ of all topologies on X which contain $\cup_{i \in I} \tau_i$ is non-empty, as the discrete topology $\mathcal{P}(X)$ contains all subsets of X. If we form the intersection $\tau = \cap_{\sigma \in \mathcal{C}} \sigma$, we see by the previous part that it is a topology on X. As $\cup_{i \in I} \tau_i \subset \sigma$ for all $\sigma \in \mathcal{C}$, we have $\cup_{i \in I} \tau_i \subset \tau$ as well. Thus τ is a topology on X that contains $\cup_{i \in I} \tau_i$. If ρ were another topology on X which contained $\cup_{i \in I} \tau_i$, then $\rho \in \mathcal{C}$, and hence $\tau = \cap_{\sigma \in \mathcal{C}} \sigma \subset \rho$. Thus τ is the smallest topology on X which contains $\cup_{i \in I} \tau_i$. Uniqueness is immediate, for if ζ were another smallest topology on X which contained $\cup_{i \in I} \tau_i$, then $\zeta \subset \tau$ and $\tau \subset \zeta$.
We prove that there is a largest topology contained in all the τ_i. Let $\tau = \cap_{i \in I} \tau_i$. Then τ is a topology on X as discussed. Suppose that σ is another topology on X which is contained in all the τ_i, i.e. $\sigma \tau_i$ for all $i \in I$. Then taking the intersection over $i \in I$ we obtain $\sigma \subset \cap_{i \in I} \tau_i = \tau$. Thus any topology which is contained in all of the τ_i is contained in τ, and so τ is the largest topology contained in all of the τ_i. The uniqueness is the same as in the previous case; if ζ were another largest topology contained in all the τ_i, then $\zeta \subset \tau$ and $\tau \subset \zeta$.

(c) If $X = \{a, b, c\}$ let $\tau_1 = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\emptyset, X, \{a\}, \{b, c\}\}$. Find the smallest topology containing τ_1 and τ_2 and the largest topology contained in τ_1 and τ_2.
The smallest topology containing both is $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Checking that τ is a topology is tedious but simple, and $\tau_1, \tau_2 \subset \tau$. If σ is a topology on X containing τ_1 and τ_2, then it must contain $\tau_1 \cup \tau_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{b, c\}\}$. As σ is a topology it must also contain $\{a, b\} \cap \{b, c\} = \{b\}$. Thus $\tau \subset \sigma$, and τ is the smallest topology containing τ_1 and τ_2.
The largest topology contained in both τ_1 and τ_2 is $\tau = \{\emptyset, X, \{a\}\}$, for this is just $\tau_1 \cap \tau_2$.

Problem 13.6
Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.
First we show that $\mathbb{R}_\ell$ does not have a finer topology that $\mathbb{R}_K$. The set $U = (-1, 1) - \{\frac{1}{n} : n \in \mathbb{N}\}$ is open in $\mathbb{R}_K$, as it is a basis element. However, this set is not open in $\mathbb{R}_\ell$, as any half-open interval $[a, b)$ which contains 0 also contains a small half-open interval of the form $[0, \epsilon)$, where $\epsilon > 0$. Such a set must contain elements of $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Thus $(-1, 1) - K$ is not open in the lower limit topology, so that the lower limit topology is not finer than the K-topology.
Now we show that the K-topology is not finer than the lower limit topology. Let $U = [0, 1)$. Then U is open in $\mathbb{R}_\ell$ as it is a basis element. But $[0, 1)$ is not open in the K-topology, as a basis element of the K-topology looks like (a, b) or $(a, b) - K$, and if either of these contains 0 it also contains $(-\epsilon, 0]$ for some $\epsilon > 0$. Thus we cannot find an open set in $\mathbb{R}_K$ which contains 0 and is contained in $[0, 1)$, so that the latter set is not open in $\mathbb{R}_K$.
Problem 13.7 Consider the following topologies on $\mathbb{R}$: τ_1 is the standard topology, τ_2 is the K-topology, τ_3 is the finite complement (cofinite) topology, τ_4 is the upper limit topology generated by the basis of all sets of the form $(a, b]$, and τ_5 is the ray topology generated by the left-infinite open rays $(-\infty, a)$ as a basis. Determine the inclusion relations amongst this

collection of topologies.
The standard topology is strictly contained in the K-topology, as we proved in class. Thus $\tau_1 \subsetneq \tau_2$.

If a subset of $\mathbb{R}$ has finite complement, it looks like $(-\infty, a_1) \cup (a_1, a_2) \cup \ldots \cup (a_{k-1}, a_k) \cup (a_k, \infty)$. This is an open set in the standard topology, so the standard topology strictly contains the finite complement topology, as $(0, 1)$ is open in the standard topology but not the finite complement topology. Thus $\tau_3 \subsetneq \tau_1$.

If (a, b) is a basis element of the standard topology and $x \in (a, b)$, then $(a, x]$ is a basis element of the upper limit topology which contains x and is contained in (a, b). Thus (a, b) is open in the upper limit topology, so that the upper limit topology contains the standard topology.

This containment is strict as $(0, 1]$ is not open in the standard topology. Thus $\tau_1 \subsetneq \tau_4$.

The ray topology is generated by a basis of sets which are all open in the standard topology on $\mathbb{R}$, hence the standard topology contains the ray topology. The set $(0, 1)$ is not open in the ray topology because no ray lies within $(0, 1)$. Hence this containment is strict. So $\tau_5 \subsetneq \tau_1$.
The K-topology strictly contains the standard topology, which strictly contains the cofinite topology. Hence the K-topology strictly contains the cofinite topology. Thus $\tau_3 \subsetneq \tau_2$.

The set $(-1, 0]$ is open in the upper limit topology but not in the K-topology for the same reason discussed in problem 6, so the K-topology does not contain the upper limit topology. So $\tau_4 \not\subset \tau_2$.

As the standard topology strictly contains the ray topology, so too does the K-topology. Thus $\tau_5 \subsetneq \tau_2$.
The cofinite topology does not contain the upper limit topology, as $(0, 1]$ is not open in the cofinite topology. The cofinite topology does not contain the ray topology as $(-\infty, 0)$ is not open in the cofinite topology.
The upper limit topology strictly contains the standard topology, which strictly contains the ray topology. Hence the upper limit topology strictly contains the ray topology. So $\tau_5 \subsetneq \tau_1 \subsetneq \tau_4$.

The upper limit topology also contains the K-topology. For let $U = (a, b) - K$ be a basis element of the K-topology (all of the non-deleted open intervals are open in the upper limit topology by an argument similar to the above). If $x \in U$ and $x \leq 0$, then $(a, x]$ is a basis element in the upper limit topology which satisfies $x \in (a, x] \in U$. If $x \in U$ and $x > 0$, let n be the least integer such that $\frac{1}{n} < x$. Then $U \cap (1/n, x] = (y, x]$, where $y = 1/n$ if $1/n > a$ and $y = a$ if $a > 1/n$. Thus $x \in (y, x] \subset (a, b) - K$, and $(x, y]$ is a basis element for the upper limit topology. Thus the set U is open in the upper limit topology. So $\tau_2 \subset \tau_4$.
Putting this all together, we have τ_5 and τ_3 not comparable with one another, but both strictly contained in τ_1, and then the rest of the containments:

$$\tau_3, \tau_5 \subsetneq \tau_1 \subset \tau_2 \subset \tau_4.$$

Problem 16.1 Show that if Y is a subspace of X, and A is a subset of Y, then the topology that A inherits as a subspace of Y is the same as the topology that A inherits as a subspace of X.
Suppose that $U \subset A$ is open in the subsace topology inherited from Y. Then there is a set $V \subset Y$, open in the subspace topology on Y, which satisfies $U = V \cap A$. As V is open in the subspace topology on Y, there is an open set $W \subset X$ which satsifies $V = W \cap Y$. Then $U(W \cap Y) \cap A = W \cap (Y \cap A) = W \cap A$, and U is the intersection of an open subset of X

with A. Thus U is open in the subspace topology inherited from X.
Suppose that $U \subset A$ is open in the subspace topology inherited from X. Then there exists an open set $W \subset X$ with $U = W \cap A$. Let $V = W \cap Y$. Then V is an open subset of Y in the subspace topology that Y inherits from X, and $V \cap A = (W \cap Y) \cap A = W \cap A = U$. So U is an open subset of A in the subspace topology inherited from Y.
Problem 16.2 Let τ and τ' be topologies on X, and suppose that τ' is strictly finer than τ. What can you say about the corresponding subspace topologies on $Y \subset X$.
You can only say that the subspace topology $\tau_Y = \{U \cap Y : U \in \tau\}$ is contained in the subspace topology $\tau'_Y = \{U \cap Y : U \in \tau'\}$. For if $U \cap Y \in \tau_Y$, then $U \in \tau$, hence $U \in \tau'$, and $U \cap Y \in \tau'_Y$. The strictness fails because one can consider the standard topology and the K-topology on the real line $\mathbb{R}$, then let $Y = \{0\}$. Both of the subspace topologies are the same.
Problem 16.4 A map $f : X \to Y$ is said to be an open map if it maps open sets in X to open sets in Y, i.e. if $U \subset X$ is open in X, then $f(U) \subset Y$ is open in Y. Show that the projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are both open maps.

Proof. We show that π_1 is open; the proof for π_2 is the mirror image. There are two approaches.
First, suppose that $W \subset X \times Y$ is an open subset of $X \times Y$. Then considering the basis of sets of the form $U \times V$ on the product topology, where $U \subset X$ and $V \subset Y$ are open in their respective topologies, there exists a collection $\{U_i \times V_i\}_{i \in I}$ such that $W = \cup_{i \in I} U_i \times V_i$. Claim: $\pi_1(W) = \cup_{i \in I} \pi_1(U_i \times V_i)$. This is a general fact about images: the image of a union of subsets of the domain of a function is the union of the images of the subsets. The proof is straightforward. The image $\pi_1(U_i \times V_i) = U_i$ as long as V_i is non-empty, which case we can safely omit by throwing out all the i such that V_i is non-empty. Thus $\pi_1(W) = \cup_{i \in I} U_i$, and as all the sets U_i are open in X, the image $\pi_1(W)$ is open in X.
The second approach: suppose that $x \in \pi_1(W)$. We find an open subset $U \subset X$ such that $x \in U$ and $U \subset \pi_1(W)$, which shows that $\pi_1(W)$ is open by the first problem. As $x \in \pi_1(W)$, there exists $(x, y) \in W$ for some $y \in Y$. As W is open, there exists a basis element $U \times V$ such that $(x, y) \in U \times V \subset W$, with $U \subset X$ and $V \subset Y$ both open in their respective topologies. Then $x \in \pi_1(U \times V) \subset \pi_1(W)$. But $\pi_1(U \times V) = U$, as V must be non-empty. Thus $x \in U \subset \pi_1(W)$, and U is an open subset of X. So $\pi_1(W)$ is an open subset of X. □

Problem 13.8 If L is a straight line in the plane, describe the topology that L inherits as a subspace of $\mathbb{R}_\ell \times \mathbb{R}$ and as a subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.
First we equip the plane with the product topology $\mathbb{R}_\ell \times \mathbb{R}$. A basis for the product topology is given by all sets of the form $[a, b) \times (c, d)$, which sets look like rectangles with all but the interior part of the left edge removed from the boundary. Let L be a straight line in the plane.

Case 1: L is horizontal or diagonal (positive or negative slope). If we intersect $[a, b) \times (c, d)$ with a non-vertical line L, we obtain either an open interval in L or a half-open interval (including left endpoint) in L. By drawing a suitable basis element B you can obtain any half-open interval in L as the intersection $B \cap L$. Thus the subspace topology on a non-vertical line L is the lower limit topology.
Case 2: If L is a vertical line such as the y-axis, the intersection with any basis element $[a, b) \times (c, d)$ is just an open interval in the line (or else empty). Any open interval in the line L can be written as the intersection $B \cap L$ for a suitably drawn basis element B. Thus

the subspace topology on L is given by the standard topology.
Now we consider the plane as $\mathbb{R}_\ell \times \mathbb{R}_\ell$. Then the basis elements for the product topology have the form $[a, b) \times [c, d)$. These look like rectangles with the upper and rightmost boundary edges removed along, including the endpoints of these edges. Case 1: The line is vertical or has non-negative slope. If we intersect such a basis element with a such a line L, we obtain a half-open interval (including left endpoint, or bottom endpoint for a vertical line). Any half-open interval in L can be written as $B \cap L$ for suitably chosen basis element B. Thus such a line L inherits the lower limit topology.

Case 2: The line has negative slope. If L is a line with negative slope, then any point $(x, y) \in L$ is also in $B = [x, x+1) \times [y, y+1)$. The intersection $L \cap B$ is just $\{(x, y)\}$, because the slope of the line connecting (x, y) to any point in $[x, x+1) \times [y, y+1)$ is positive (hence not contained in L). Thus L inherits the discrete topology.