MATH 54 - HW SOLUTIONS 3

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Problem 17.1

This is an application of DeMorgan's laws.

(1) $X, \emptyset$ are both in τ, as their complements are each contained in $\mathcal{C}$.
(2) Supposing that $\{U_i\}_{i\in I}$ is a family of sets from τ, the complement $X - \cup_{i\in I} U_i$ equals $\cap_{i\in I}(X - U_i)$ by DeMorgan's laws. Note that each complement $X - U_i$ is an element of $\mathcal{C}$, as each $U_i \in \tau$. As arbitrary intersections of elements from $\mathcal{C}$ are contained in $\mathcal{C}$, we have $X - \cup_{i\in I} U_i \in \mathcal{C}$. Thus $\cup_{i\in I} U_i \in \tau$. So τ contains the union of any collection of its elements.
(3) Suppose that $U_1, \dots, U_n$ is a finite collection of elements from τ. Then the complement $X - \cap_{i=1}^n U_i$ equals $\cup_{i=1}^n X - U_i$ by DeMorgan's laws. As each $U_i \in \tau$, each $X - U_i \in \mathcal{C}$. As $\mathcal{C}$ contains finite unions of its elements, we have $X - \cap_{i=1}^n U_i \in \mathcal{C}$. Thus $\cap_{i=1}^n U_i \in \tau$. So τ contains the union of any finite collection of its elements.

Problem 17.3

Suppose that $A \subset X$ is closed and $B \subset Y$ is closed. We show that $A \times B \subset X \times Y$ is closed, by proving that the complement $X \times Y - A \times B$ is open.

Let $(x, y) \in X \times Y - A \times B$. Then $x \notin A$ or $y \notin B$. If $x \notin A$, then $x \in X - A$. As A is closed, $X - A$ is open, and so $X - A \times Y$ is a basis element containing (x, y). Also, we have $X - A \times Y \subset X \times Y - A \times B$, for any element of $X - A \times Y$ cannot lie in $A \times B$. Thus about an arbitrary element (x, y) of the set $X \times Y - A \times B$, we have produced a basis element $X - A \times Y$ which satisfies $(x, y) \in X - A \times Y \subset X \times Y - A \times B$.

Problem 17.6

Let A,B, and A_i denote subsets of a space X.

(1) If $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proof. There are several proofs possible. We give two. Suppose that $A \subset B$ and suppose $x \in \overline{A}$. If U is a nbhd of x, then there exists $a \in U \cap A$. As $A \subset B$, we have $a \in U \cap B$. Thus every nbhd of x contains a point of B, and we conclude that $x \in \overline{B}$.

Suppose that $A \subset B$. Note that every closed set $F \subset X$ which contains B also contains A. Thus on forming the closure of B, we take the intersection over a smaller collection of closed subsets of X than that collection whose intersection we take to form the closure of A. Everything which lies in the intersection of the collection of closed sets containing A also lies in every closed set containing B. Thus $\overline{A} = \cap_{F:A\subset F, F=\overline{F}} F \subset \cap_{F':B\subset F', F'=\overline{F'}} F'$.

□

(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. Note that $A \subset \overline{A}$ and $B \subset \overline{B}$, so that $A \cup B \subset \overline{A} \cup \overline{B}$. The latter set is a closed set containing $A \cup B$, as the binary union of closed sets is closed. Thus we have $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$.

For the other direction, notice that $A \subset A \cup B$, so that $\overline{A} \subset \overline{A \cup B}$. Similarly, $\overline{B} \subset \overline{A \cup B}$. Taking unions of both inclusions we obtain $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

Thus we have shown both inclusions and the equation $\overline{A} \cup \overline{B} = \overline{A \cup B}$ holds. □

(3) $\overline{\cup_i A_i} \supset \cup \overline{A_i}$; give an example where equality fails.

Proof. For each i, we have $A_i \subset \cup_i A_i$. Thus $\overline{A_i} \subset \overline{\cup_i A_i}$. Taking the union over all such inclusions we obtain $\cup_i \overline{A_i} \subset \overline{\cup_i A_i}$.

Let $X = \mathbb{R}$, let $I = (0, 1]$ and let $A_i = \{i\}$ for $i \in I$. Then $\overline{A_i} = \{i\}$ for all $i \in I$. So $\cup_{i \in I} \overline{A_i} = (0, 1]$ Yet $\cup_{i \in I} A_i = (0, 1]$, and so $\overline{\cup_{i \in I} A_i} = \overline{(0, 1]} = [0, 1]$. □

Problem 17.7: The proof fails at "so that x must belong to the closure of some A_α". For it is not true that every nbhd U of x intersects the same A_α – only that for each U there is one A_α which intersects U.

Problem 17.8: Let A, B, A_i denote subsets of a space X.

(1) The only inclusion that holds is $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. For $A \cap B \subset A \subset \overline{A}$ and $A \cap B \subset B \subset \overline{B}$. Thus $\overline{A} \cap \overline{B}$ is a closed containing $A \cap B$. Hence $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$.

(Alternatively, let $x \in \overline{A \cap B}$. Then every nbhd of x contains a point of $A \cap B$, which in particular is a point of A and a point of B. Thus $x \in \overline{A}$ and $x \in \overline{B}$).

If we let $A = (-1, 0)$ and $B = (0, 1)$ as subsets of $\mathbb{R}$, then $A \cap B = \emptyset$, whence $\overline{A \cap B} = \emptyset$. But $\overline{A} = [-1, 0]$ and $\overline{B} = [0, 1]$, so that $\overline{A} \cap \overline{B} = \{0\}$.

(2) Again, the inclusion that holds is $\overline{\cap_i A_i} \subset \cap_i \overline{A_i}$. Notice that $A_i \subset \overline{A_i}$, so that $\cap_i A_i \subset \cap_i \overline{A_i}$. But the latter set is closed, as it is the finite intersection of a collection of closed sets. Thus it is a closed set containing $\cap_i A_i$, and so it contains the closure of $\cap_i A_i$, i.e. $\overline{\cap_i A_i} \subset \cap_i \overline{A_i}$.

The same counterexample as in part 1 demonstrates why the reverse inclusion does not hold.

(3) The only inclusion that holds is $\overline{A} - \overline{B} \subset \overline{A - B}$. For if $x \in \overline{A} - \overline{B}$, then every nbhd of x contains a point of A and there is some nbhd V of x which contains no point of B. So let U be an arbitrary nbhd of x. Then $U \cap V$ is also a nbhd of x, as the intersection of two open sets containing x is also an open set containing x. But notice that $U \cap V \cap B \subset V \cap B = \emptyset$, so that $U \cap V$ contains no points of B. But $U \cap V$ must contain a point $a \in A$, as it is a nbhd of $x \in \overline{A}$. As $a \notin B$, we have $a \in U \cap V \cap (A - B) \subset U \cap (A - B)$. Thus every nbhd about x contains a point of $A - B$.

To see why the reverse inclusion does not hold, let $A = [0, 1]$ and $B = (0, 1]$ as subsets of $\mathbb{R}$. Then $A - B = \{0\}$, which has closure $\overline{\{0\}}$. But $\overline{A} = [0, 1] = \overline{B}$, so that the complement of the closures is $\emptyset$. Thus $\overline{A - B} \not\subset \overline{A} - \overline{B}$.

Problem 17.9

Let $A \subset X$ and $B \subset Y$. Show that in $X \times Y$ the equation $\overline{A} \times \overline{B} = \overline{A \times B}$.

Proof. Notice that as $A \subset \overline{A}$ and $B \subset \overline{B}$, we have $A \times B \subset \overline{A} \times \overline{B}$. By problem 17.3, the latter set is closed in $X \times Y$. Thus $\overline{A \times B} \subset \overline{A} \times \overline{B}$.

Now suppose that $(x, y) \in \overline{A} \times \overline{B}$, so that $x \in \overline{A}$ and $y \in \overline{B}$. Let $U \times V$ be a basis element for $X \times Y$ which contains (x, y). Then U is a nbhd of x, so that U contains a point $a \in A$, and similarly V contains a point $b \in B$. Thus $(a, b) \in U \times V \cap A \times B$. So every basis element which contains (x, y) also contains a point of $A \times B$. Thus $(x, y) \in \overline{A \times B}$, and $\overline{A} \times \overline{B} \subset \overline{A \times B}$

As both inclusions hold, the equation $\overline{A} \times \overline{B} = \overline{A \times B}$. □

Problem 17.10

Show that every order topology is Hausdorff.

Proof. Suppose that X is an ordered set and $x < y$ in X. We construct disjoint nbhds of x and y respectively. If x is the least element and y is the greatest element of X, then $[x, y)$ and $(x, y]$ are disjoint basis elements in the order topology on X which contain the pts x and y respectively.

If x is the least element and y is not the greatest element, pick an element z which is greater than y. Now two sub-cases arise: If there is no element in (x, y), the sets $[x, y)$ and (x, z) are the required nbhds. If there is an element $u \in (x, y)$, the set $[x, u)$ and (u, z) are the required nbhds.

If x is not the least element but y is the greatest element, we pick an element $w < x$. Then (w, y) and $(x, y]$ are disjoint nbhds if $(x, y) = \emptyset$, and if $u \in (x, y)$ then (w, u) and $(u, y]$ are the required disjoint nbhds.

If x is not the least element and y is not the greatest element, say $w < x < y < z$, then we again have two cases. If $(x, y) = \emptyset$, then (w, y) and (x, z) are the required sets. If $u \in (x, y)$ then (w, u) and (u, z) are the required sets.

□

Problem 17.11

Show that the product of two Hausdorff spaces is Hausdorff.

Proof. Suppose that X and Y are Hausdorff spaces. Let (x, y) and (x', y') be distinct points of $X \times Y$. Assume that $x \neq x'$ (the other case is handled similarly). As X is a Hausdorff space, there are nbhds $U, V \subset X$ of x and x' such that $U \cap V = \emptyset$. Then $U \times Y$ and $V \times Y$ are both basis elements in $X \times Y$, hence open. But $U \cap V = \emptyset$, so that $U \times Y \cap V \times Y = \emptyset$, as no point of x can serve as the first coordinate of a point in the intersection $U \times Y \cap V \times Y$. As $(x, y) \in U \times Y$ and $(x', y') \in V \times Y$, we have that $X \times Y$ is Hausdorff. (The case when $x = x'$ but $y \neq y'$ is handled similarly). □

Problem 17.13

Show that X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is closed in $X \times X$.

Proof. Suppose that X is Hausdorff. We show that the complement of the diagonal is open. Let $(x,y) \in X \times X - \Delta$. Then $X \neq y$, so there exist disjoint nbhds U of x and V of y, respectively. Then $U \times V$ is a nbhd of (x,y) in the product topology. If $(z,z) \in U \times V$, then $z \in U \cap V$, which violates $U \cap V = \emptyset$. Thus $U \times V \subset X \times \Delta$. Thus about any point (x,y) of $X \times Y$ we have a produced a basis element $U \times V$ which is contained in $X \times Y - \Delta$. Thus the complement of Δ is open, and Δ is closed.

Now suppose that Δ is closed in $X \times Y$. Suppose that $x \neq y$ are two points of X. Then (x,y) lies in the complement of Δ. As this is an open set, we can find a basis element $U \times V$ in the product space $X \times Y$ such that $(x,y) \in U \times V \subset X \times Y - \Delta$. But the condition that $(U \times V) \cap \Delta = \emptyset$ precisely guarantees that the nbhds U and V of x and y are disjoint. Thus X is Hausdorff. $\square$