MATH 54 - HW SOLUTIONS 4

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Problem 18.2: No, it is not necessary. The constant function $f : [0,1] \to \{0\}$ sends the limit point $\{1\}$ of $[0,1]$ to 0, which is not a limit point of the image.

Problem 18.3:

(a) The map i is continuous if and only if $i^{-1}(U)$ is in τ' for any $U \in \tau$¿ if and only if U is in τ' for any $U \in \tau$ if and only if $\tau \subset \tau'$.
(b) The map i is a homeomorphism if and only if $i : X' \to X$ and $i : X \to X'$ are both continuous if and only if $\tau' \subset \tau$ and $\tau \subset \tau'$ if and only if $\tau' = \tau$.

Problem 18.4 Let $y_0 \in Y$, let $f : X \to X \times Y$ be given by $f(x) = x \times y$. The image of f is $f(X) = X \times \{y_0\}$. If we restrict the codomain to the range, we obtain a bijective function $f : X \to X \times \{y_0\}$. As the projections $f(x) = x$ and $f(x) = y_0$ onto each coordinate are continuous, the function is continuous by the universal mapping property. For the function is the map produced by the universal mapping property, which is always continuous. We show that $f^{-1} : X \times \{y_0\} \to X$ given by $f(x \times y_0) = x$ is continuous. Suppose that $U \subset X$ is an open nbhd of $x = f^{-1}(x \times y_0)$. Then the preimage of U under f^{-1} is $(f^{-1})^{-1}(U) = f(U) = U \times \{y_0\}$. This is an open subset of the image $f(X)$ (which has the subspace topology), for $U \times \{y_0\} = (U \times Y) \cap f(X)$, and $U \times Y$ is open in $X \times Y$. Thus the inverse function f^{-1} is continuous and f is a homeomorphism onto its range.

Problem 18.5 Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \frac{x-a}{b-a}$. The preimage of the open interval (c,d) under f is $(c(b-a)+a, d(b-a)+a)$ also an open interval. Hence the preimage of a basis element is a open, and f is continuous. The inverse map f^{-1} is $f^{-1}(x) = (b-a)x + a$, which is continuous for the same reason. Thus f and f^{-1} are both continuous. The restriction of f to any subset $A \subset \mathbb{R}$ therefore yields a homeomorphism $f|_A : A \to f(A)$, by Theorem 18.2. In particular, $f : (a,b) \to (0,1)$ is a homeomorphism as is $f : [a,b] \to [0,1]$.

Problem 18.7a

(a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right. We show that it is continuous from $\mathbb{R}$ to $\mathbb{R}_\ell$. Let $a \in \mathbb{R}$, let $\epsilon > 0$. We construct a nbhd of a in $\mathbb{R}_\ell$ which maps into $(f(a) - \epsilon, f(a) + \epsilon)$. As f is continuous from the right, there exists δ such that if $a \le x < a + \delta$, then $|f(x) - f(a)| < \epsilon$. Then $[a, a + \delta)$ is nbhd of a in $\mathbb{R}_\ell$ which maps into $(f(a) - \epsilon, f(a) + \epsilon)$. Thus f is continuous from $\mathbb{R}_\ell$ to $\mathbb{R}$.
(b) The only continuous maps from $\mathbb{R}$ to $\mathbb{R}_\ell$ are constant maps. If a function f is continuous from $\mathbb{R}_\ell$ to $\mathbb{R}_\ell$, it is also continuous from $\mathbb{R}_\ell$ to $\mathbb{R}$ (fewer open sets in the codomain). You can show this implies that f is right continuous.

Problem 18.8

(a) Let $A = \{x | f(x) \leq g(x)\}$. We show that $X - A$ is open.

Suppose x is a point of $X - A$, so that $f(x) > g(x)$. As order topologies are Hausdorff, we can pick open nbhds U and V of $f(x)$ and $g(x)$ respectively such that $U \cap V = \emptyset$. As the order topology is generated by open and half open intervals, we can find two (open or half-open) intervals P and Q in Y such that $f(x) \in P$ and $g(x) \in Q$ and $I_1 \cap I_2 = \emptyset$.

Claim: if $p \in P$ and $q \in Q$, then $q < p$. For otherwise there exist $p \in P$ and $q \in Q$ such that $p < q$ (they cannot be equal as the intervals are disjoint). If $q < g(x)$, then $q < f(x)$, and we have $p < q < f(x)$, which gives $q \in P$ (as an interval contains all intervals between its points). This contradicts $P \cap Q = \emptyset$, so $g(x) < q$. If $f(x) < q$, then $p < g(x) < f(x)$, so that $g(x) \in P$, a contradiction. Thus $q < f(x)$. So $p < q < f(x)$, which forces $q \in P$, a contradiction. We have $f(x) \geq q$ and $q \geq f(x)$, which forces $f(x) = q$, a contradiction. Thus we cannot have $p \in P$ and $q \in Q$ with $p < q$. (This proof is a little tedious – you can convince yourself that two disjoint intervals are ordered with respect to one another by sketching two intervals in the real line).

We have $x \in f^{-1}(P) \cap g^{-1}(Q)$, a nbhd of x as each interval is open and the maps f and g are continuous. If $z \in f^{-1}(P) \cap g^{-1}(Q)$, then $f(z) \in P$ and $g(z) \in Q$. Hence $f(z) > g(z)$, and $z \in X - A$. Thus about a point of $X - A$ we have found a nbhd which is contained in $X - A$.

(b) We can write $X = C_1 \cup C_2$, where $C_1 = \{x : f(x) \leq g(x)\}$ and $C_2 = \{x : g(x) \leq f(x)\}$. Then C_1 and C_2 are both closed, with $h|_{C_1} = f$ and $h|_{C_2} = g$. Thus each restriction is continuous. The two functions agree on the intersection $C_1 \cap C_2$, because if $z \in C_1 \cap C_2$, then $f(x) \leq g(x) \leq f(x)$, which compels $f(x) = g(x)$. Thus the domains are closed sets, the restrictions are continuous, and the functions agree on the overlap. So h is continuous by the pasting lemma.

Problem 18.10 We show that the preimage of a basis element $U \times V$ in $B \times D$ is open, where $U \subset B$ is open and $V \subset D$ is open.

Note that $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$. Proof: $(f \times g)(x \times y) \in U \times V$ if and only if $f(x) \in U$ and $g(y) \in V$ if and only if $x \in f^{-1}(U)$ and $y \in g^{-1}(V)$. If U and V are open, and the maps f and g are continuous, then $f^{-1}(U)$ and $g^{-1}(V)$ are both open. Hence the product $f^{-1}(U) \times g^{-1}(V)$ is a basis element in the product topology on $A \times C$, hence open. Thus $f \times g$ is a continuous map.

Problem 18.12

(a) We use the fact that continuous functions from calculus are continuous. If we fix x or y to be nonzero, we obtain a function $F(y) = \frac{ay}{a^2+y^2}$ or $F(y) = bxb^2 + x^2$. These are continuous as they are quotients of continuous functions. If x or y is fixed to be zero, then we obtain the zero function, which is always continuous.

(b) We compute: $g(x) = F(x \times x) = \begin{cases} 1/2 & x \neq 0. \\ 0 & x = 0. \end{cases}$.

(c) This is not continuous from $\mathbb{R}$ to $\mathbb{R}$ because the pre-image of the open set $(-1/2, 1/2)$ is $\{0\}$, which is not open in $\mathbb{R}$.

Problem 19.1 (Box Topology): Each of these sets is a basis element in the box topology. We show that each basis element in the box topology is open in the topology generated by these sets. Let $\prod_\alpha U_\alpha$ be a basis element in the product topology. If $x \in \prod_\alpha U_\alpha$, then $x(\alpha) \in U_\alpha$ for each α. As $\mathcal{B}_\alpha$ is a basis for X_α, we can produce $B_\alpha \in \mathcal{B}_\alpha$ such that $x_\alpha \in B_\alpha \subset U_\alpha$. Then $x \in \prod B_\alpha \subset \prod U_\alpha$. Thus $\prod U_\alpha$ is open in the topology generated by the sets $\prod B_\alpha$. (Product Topology): Each of these sets is a basis element in the product topology. We show that a basis element $\prod U_\alpha$, where all but finitely many $U_\alpha = X_\alpha$, is open in the topology generated by these sets. Let $x \in \prod U_\alpha$. Then for the finitely many $U_{\alpha_1}, \ldots, U_{\alpha_n}$ which do not equal their respective X_{α_i}, we select basis elements $B_{\alpha_1}, \ldots, B_{\alpha_n}$ such that $x_{\alpha_i} \in B_{\alpha_i} \subset U_{\alpha_i}$. Then the set $\prod_\alpha B_\alpha$, where $B_\alpha = B_{\alpha_i}$ if α is α_i, and $B_\alpha = X_\alpha$ if α is not one of the finitely many α_i, is one of the proposed basis elements, and we have $x \in \prod B_\alpha \subset \prod U_\alpha$. Thus $\prod U_\alpha$ is open in the topology generated by the sets, and the sets form a basis.

Problem 19.3 We show that, given each X_α is Hausdorff, the space $\prod X_\alpha$ is Hausdorff in the product topology. As the box topology is finer, this proves that the space is Hausdorff in the box topology as well. Let $x, y \in \prod X_\alpha$ be distinct. Then there exists some index β such that $x(\beta) \neq y(\beta)$. As X_β is a Hausdorff space, we can pick open subsets U_β and V_β of X_β such that $x(\beta) \in U_\beta$, $y(\beta) \in V_\beta$ and $U_\beta \cap V_\beta = \emptyset$. Then let $U = \prod U_\alpha$ where $U_\alpha = X_\alpha$ if $\alpha \neq \beta$ and $U_\alpha = U_\beta$ if $\alpha = \beta$. Similarly let $V = \prod V_\alpha$, where $V_\alpha = X_\alpha$ if $\alpha \neq \beta$ and $V_\beta = V_\beta$ constructed above. Then $x \in U$ and $y \in V$, and these basis elements of the product topology are disjoint. For anything in U has its βth entry in U_β and anything in V has its βth entry in V_β, and these two sets are disjoint.

Problem 19.6 Suppose that the sequence $x_1, \ldots$ converges to $x \in \prod X_\alpha$. We show that for each α, $\pi_\alpha(x_n)$ converges to $\pi_\alpha(x)$. Let U be a nbhd of $\pi_\alpha(x)$. Then $\pi_\alpha^{-1}(U)$ is a nbhd of x in $\prod X_\alpha$. Hence there is $N > 0$ such that if $n > N$, then $x_n \in \pi_\alpha^{-1}(U)$. Thus if $n > N$, $\pi_\alpha(x_n) \in U$. Thus the sequence $\pi_\alpha(x_n)$ converges to $\pi_\alpha(x)$.

Suppose that for all α, the sequence $\pi_\alpha(x_n)$ converges to $\pi_\alpha(x)$. We show that x_n converges to x. Let U be a nbhd of x. We can take a basis element $\prod U_\alpha$ of $\prod X_\alpha$ such that $x \in \prod U_\alpha \subset U$, and $U_\alpha = X_\alpha$ for all but finitely many α_i, $i = 1, \ldots, k$.

The finitely many U_{α_i} are nbhds of the respective $\pi_{\alpha_i}(x)$. We can find N_i such that $n > N_i$ implies that $\pi_{\alpha_i}(x) \in U_{\alpha_i}$.

Let $N = \max\{N_1, \ldots, N_k\}$. We claim that if $n > N$, then $x_n \in \prod U_\alpha = \cap_{i=1}^k \pi_{\alpha_i}^{-1}(U_{\alpha_i})$. For if $n > N$, then for each α_i we have $n > N_i$, so that $\pi_{\alpha_i}(x_n) \in U_{\alpha_i}$. Thus $x_n \in \pi_{\alpha_i}^{-1}(U_{\alpha_i})$. So $x_n \in U$ for all $n > N$, and $x_n \to x$.

Problem 19.10 Let A be a set, let $\{X_i\}$ be an indexed family of spaces, let $\{f_i\}$ be an indexed family of maps $f_i : A \to X_i$.

(a) There is at least one topology on A such that all of the maps f_i are continuous, namely the discrete topology. For the preimage of any set $f_i^{-1}(Y_i)$, where $Y_i \subset X_i$, is a subset of A, and hence open in the discete topology. Thus we can consider $F = \{\sigma :$ each map f_i is continuous if A is given the topology $\sigma\}$. If we form $\tau = \cap_{\sigma\in F}\sigma$, then τ is also a topology on X, contained in every topology $\sigma \in F$. Suppose that $U_i \subset X_i$ is open. Then $f_i^{-1}(U_i) \in \sigma$ for all $\sigma \in F$. Thus $f_i^{-1}(U_i) \in \tau$. Thus all the maps f_i are continuous if you equip A with the topology τ, and τ is therefore the smallest topology on A which makes all the maps f_i continuous.

(b) All of the sets $f_i^{-1}(U_i)$ are open in τ, as they are the preimages of open sets under the maps f_i. Thus $\mathcal{S} \subset \tau$. Thus the topology generated by the sub-basis is no larger than τ. We show that the topology generated by the sub-basis $\mathcal{S}$ makes all of the maps f_i continuous. Let $U_i \subset X_i$ be open. Then $f_i^{-1}(U_i) \in \mathcal{S}$, so that the maps f_i are continuous in the topology generated by the sub-basis $\mathcal{S}$. Thus the topology is given by the sub-basis $\mathcal{S}$.

(c) Suppose that each $f_i \circ g$ is continuous. We show that the preimage under g of a sub-basis element $f_i^{-1}(U_i) \in \mathcal{S}$, where U_i is an open subset of X_i, is an open subset of Y. We have the rule $g^{-1}(f_i^{-1}(U_i)) = (f_i \circ g)^{-1}(U_i)$ for preimages. As each composition $f_i \circ g$ is continuous, and U_i is open in X_i, the latter set in the equation of preimages is open. Thus the preimage under g of a sub-basis element is open. As open sets in A (with the topology τ) are simply arbitrary unions of finite intersections of sub-basis elements, the preimage under g of any open set in A is open in Y. Thus g is continuous.

Suppose that g is continuous. As each map $f_i : A \to X_i$ is continuous when you equip A with the topology τ, the composition $f_i \circ g : Y \to X_i$ is continuous as it is a composition of continuous functions.

(d) Suppose that $U \in \tau$. Let $z \in f(U)$. We produce an open set $V \subset \prod X_\alpha$ such that $V \cap f(A) \subset f(U)$ and $z \in V$. Note that this shows that $f(U)$ is open in the subspace topology on $f(A)$. Pick $x \in U$ such that $f(x) = z$. We can find finitely many $i_1, \dots, i_m$ and open sets $U_{i_k} \subset X_{i_k}$ such that $x \in f_{i_k}^{-1}(U_{i_k})$ and $\cap_{k=1}^m f_{i_k}^{-1}(U_{i_k}) \subset U$.

Let $V = \prod_i U_i$, where $U_i = X_i$ for indices i not on the list $i_1, \dots, i_m$ and $U_{i_k} = U_{i_k}$ for indices i_k on the list. Then $z \in V$, for $z_{i_k} = f_{i_k}(x) \in U_{i_k}$ for the indices i_k on the list, and $z_i \in U_i = X_i$ for indices i not on the list. As V is a basis element in the product topology, it is open.

Claim: $V \cap f(A) \subset f(U)$. For suppose that $(x_i) \in V \cap f(A)$. Then $x_i \in U_i$ for all indices i. Also, $(x_i) = f(a)$ for some $a \in A$. Then $f(a)_i \in U_i$ for all i, so that $a \in f_i^{-1}(U_i)$ for all indices i. Thus $a \in \cap_{k=1}^m f_{i_k}^{-1}(U_{i_k}) \subset U$, and $z \in f(U)$.

Thus for every point $z \in f(U)$, there is an open set V in $\prod X_i$ containing z such that $z \in f(A) \cap V \subset f(U)$. As sets of the form $f(A) \cap W$, where $W \subset \prod X_i$ is open in the product topology, are the open sets in the subspace topology on $f(A)$, we have that $f(U)$ is open.