Math 54 - HW Solutions 5

Dan Crytser

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Problem 20.1a

To show that the Manhattan metric $d(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n|$ induces the standard topology on $\mathbb{R}^n$, we show that it induces the same topology as the square metric. Let $\rho(x,y)$ denote the square distance between x and y in $\mathbb{R}^n$, i.e. $\rho(x,y) = \max |x_i - y_i|$.

First we prove that $d(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n|$ defines a metric. As it is a sum of non-negative numbers for any x, y, $d(x,y)$ is always non-negative. If $d(x,y) = 0$, then $0 \leq |x_k - y_k| \leq d(x,y) = 0$ for any k. Thus $x_k = y_k$ for all k, and $x = y$. The symmetry follows from $|x_k - y_k| = |y_k - x_k|$. Let $x, y, z \in \mathbb{R}^n$. The triangle inequality follows from $|x_k - y_k| \leq |x_k - z_k| + |z_k - y_k|$, and then taking the sum over all k.

Suppose that $x, y \in \mathbb{R}^n$. Claim: $\rho(x,y) \leq d(x,y)$. For $\rho(x,y) = |x_i - y_i|$ for some index i, so we have $\rho(x,y) = |x_i - y_i| \leq |x_1 - y_1| + \ldots + |x_n - y_n| = d(x,y)$. The inequality comes from adding non-negative numbers to $|x_i - y_i|$.

Claim: $d(x,y) \leq n\rho(x,y)$. Let k be an index. Then $|x_k - y_k| \leq |x_i - y_i| = \rho(x,y)$. Then summing over $k = 1, \ldots, n$, we obtain $\sum_{k=1}^n |x_k - y_k| \leq |x_i - y_i| + |x_i - y_i| + \ldots + |x_i - y_i| = n|x_i - y_i|$.

Now we can show that the topologies generated by the square metric and the Manhattan metric are the same. Let $\epsilon > 0$ and let $x \in \mathbb{R}^n$. We show that $B_d(x,\epsilon)$ is open in the square metric. Claim: $B_\rho(x, \epsilon/n) \subset B_d(x,\epsilon)$. For if $\rho(x,y) < \epsilon/n$, then $d(x,y) \leq n\rho(x,y) < \epsilon$.

Now we show that the topology induced by the Manhattan metric contains the topology induced by the square metric. Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. We show that $B_\rho(x,\epsilon)$ is open in the Manhattan metric. Claim: $B_d(x,\epsilon) \subset B_\rho(x,\epsilon)$. For if $d(x,y) < \epsilon$, then $\rho(x,y) \leq d(x,y) < \epsilon$.

Problem 20.3 Let X be a metric space with metric d.

(a) We show that the metric map $d : X \times X \to \mathbb{R}$ is continuous. Let $x \times x' \in X \times X$. Set $C = d(x,x')$, and let $(C - \epsilon, C + \epsilon)$ be a basis element of the standard topology which contains C. Let $U = B(x, \frac{\epsilon}{2})$ and let $V = B(x', \frac{\epsilon}{2})$. Then $U \times V$ is a nbhd of $x \times x'$ in $X \times X$.

Claim: $d(U \times V) \subset (C - \epsilon, C + \epsilon)$. For if $u \in U$ and $v \in V$, then

$$d(u,v) \leq d(u,x) + d(x,v) \leq d(u,x) + d(x,y) + d(y,v) < \frac{\epsilon}{2} + C + \frac{\epsilon}{2} = C + \epsilon.$$

So $d(u,v) < C+\epsilon$. But also $d(x,y) \le d(x,u)+d(u,v)+d(v,y) < d(u,v)+\epsilon$. So we also have $C-\epsilon < d(u,v)$. Thus $u\times v \in U\times V$ implies that $d(u,v)\in (C-\epsilon, C+\epsilon)$. Thus d is continuous.

(b) Suppose that $d : X'\times X' \to \mathbb{R}$ is continous. We show that the topology from X' contains the metric topology. Let $x\in X$, let $B(x,\epsilon)$ be an open ball for some $\epsilon > 0$. We prove that there is an open set U in X' such that $x\in U\subset B(x,\epsilon)$. Note that $d(x,x)\in(-\epsilon,\epsilon)$, so that $x\times x$ is in the set $d^{-1}(-\epsilon,\epsilon)$. As d is continous when the domain is $X'\times X'$, this preimage must be a nbhd of $x\times x$. Thus we can take a basis element in the product topology $U\times V$ such that $x\times x\in U\times V\subset d^{-1}(-\epsilon,\epsilon)$. Then U is an open subset of X' which contains x. We prove that $U\subset B(x,\epsilon)$. Suppose that $y\in U$. Then $y\times x\in U\times V$, hence $d(y,x)\in(-\epsilon,\epsilon)$. But this means $y\in B(x,\epsilon)$. So $x\in U\subset B(x,\epsilon)$, and the topology on X' is at least as fine as the metric topology.

Problem 20.9 Show that the Euclidean metric on $\mathbb{R}^n$ is a metirc.

(a)

$$x\cdot(y+z)=\sum_{i=1}^{n} x_i(y_i+z_i)=\sum_{i=1}^{n}(x_iy_i+x_iz_i)=x\cdot y+x\cdot z.$$

(b) If either x or y is 0, then $|x\cdot y|=0$, so the claim holds. Suppose that both are nonzero. We compute the square norm of $\frac{x}{||x||}+\frac{y}{||y||}$, which must be greater than or equal to 0.

$$\begin{aligned} ||\frac{x}{||x||}+\frac{y}{||y||}||^2 &= \frac{x\cdot x}{||x||^2}+2\frac{x\cdot y}{||x||||y||}+\frac{y\cdot y}{||y||^2} \\ &= 2(1+\frac{x\cdot y}{||x||||y||})\ge 0 \end{aligned}$$

Now divide by 2 and subtract over the -1. We obtain $\frac{x\cdot y}{||x||||y||}\ge -1$. Multiply both sides by $-||x||||y||$ to obtain

$$-(x\cdot y)\le ||x||||y||.$$

If we repeat the process with $\frac{x}{||x||}-\frac{y}{||y||}$ and we obtain:

$$\begin{aligned} (\frac{x}{||x||}-\frac{y}{||y||})^2 &= \frac{x\cdot x}{||x||^2}-2\frac{x\cdot y}{||x||||y||}+\frac{y\cdot y}{||y||^2} \\ &= 2(1-\frac{x\cdot y}{||x||||y||}) \\ &\ge 0 \end{aligned}$$

Now we divide by 2 and add $\frac{x\cdot y}{||x||||y||}$ to both sides of the inequality to obtain $1\ge\frac{x\cdot y}{||x||||y||}$. Now multiply through by the product of the norms to obtain $x\cdot y\le ||x||||y||$. As $|x\cdot y|$ is either $x\cdot y$ or $-x\cdot y$, we have that $|x\cdot y|\le ||x||||y||$.

(c) Let $x, y \in \mathbb{R}^n$. Then

$$\begin{aligned}||x+y||^2 &= x \cdot x + 2x \cdot y + y \cdot y \\ &= ||x||^2 + 2x \cdot y + ||y||^2 \\ &\leq ||x||^2 + 2||x||||y|| + ||y||^2 \\ &= (||x|| + ||y||)^2.\end{aligned}$$

Now take square roots.

(d) Let $x, y, z \in \mathbb{R}^n$. First, notice that $d(x,y) = ||x-y|| \geq 0$. If $d(x,y) = 0$, then $||x-y|| = 0$. But as $|x_i - y_i| \leq ||x-y||$ for every $i = 1, \dots, n$, we then have $x = y$. As $(x_i - y_i)^2 = (y_i - x_i)^2$ for all i, we have that $d(x,y) = d(y,x)$ for any $x, y \in \mathbb{R}^n$.

Now

$$d(x,y) = ||x-y|| = ||(x-z)+(z-y)|| \leq ||x-z|| + ||z-y|| = d(x,z) + d(z,y).$$

Thus d is a metric.

Problem 20.11 First we check that d' is a metric, then we show that it induces the same topology as d. Let $x, y, z \in X$. Note that, as a (finite) quotient of non-negative numbers, d' is non-negative. If $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = 0$, then we must have that the numerator $d(x,y) = 0$. This compels $x = y$. If $x = y$, then $d(x,y) = 0$ which compels $d'(x,y) = 0$. Symmetry follows from the well-definition of algebraic operations (fancy talk for "duh"). The interestin part is checking that the triangle inequality holds.

We prove the hint. Note that $\frac{x}{1+x} = 1 - \frac{1}{1+x}$. Increasing x increases the denominator of $\frac{1}{1+x}$, which therefore decreases. But this increases the negative of $\frac{1}{1+x}$, so that $\frac{x}{1+x}$ is an increasing function.

$$\begin{aligned}d'(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\ &\leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\ &= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\ &\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\ &= d'(x,z) + d'(z,y)\end{aligned}$$

The first inequality comes from the triangle inequality for d and the increase of the function $\frac{x}{1+x}$ on $[0, \infty)$.

Now we check that d' and d induce the same topology on X. Let $x \in X$ and let $\epsilon > 0$. We show that $B_d(x, \epsilon)$ is open in the topology induced by d'. Let $\delta = \frac{\epsilon}{1+\epsilon}$. Claim: $B_{d'}(x, \delta) = B_d(x, \epsilon)$.

Note about increasing functions: Let $f : \mathbb{R} \to \mathbb{R}$ be increasing. If $f(x) < f(y)$, then $x < y$. For we cannot have $x = y$, and if $x > y$ we would have $f(x) > f(y)$.

Suppose that $y \in B_{d'}(x, \delta)$. Then $d'(x, y) < \delta = \frac{\epsilon}{1+\epsilon}$. As d' increases as a function of d, we must have that $d(x, y) < \epsilon$. Similarly, if $d(x, y) < \epsilon$, then $d'(x, y) = \frac{d(x,y)}{1+d(x,y)} < \frac{\epsilon}{1+\epsilon} = \delta$. Thus the open balls are equal.

Before we prove the second part, we note that algebra can be used to solve $d(x, y) = \frac{d'(x,y)}{1-d'(x,y)} = \frac{1}{1-d'(x,y)} - 1$. This gives d as an increasing function of d'.

Now we show that the ball $B_{d'}(x, \epsilon)$ is open in the topology generated by d. Let $\delta = \frac{1}{1-\epsilon} - 1$. Claim: $B_d(x, \delta) = B_{d'}(x, \epsilon)$. Suppose $y \in B_d(x, \delta)$. Then $d(x, y) < \delta$ and so $d'(x, y) = \frac{d(x,y)}{1+d(x,y)} < \epsilon$. Suppose instead that $d'(x, y) < \epsilon$. Then $d(x, y) = \frac{d'(x,y)}{1-d'(x,y)} < \frac{\epsilon}{1-\epsilon} = \delta$ (here we are assuming that $\epsilon < 1$, but this is fine– as long as balls of radius $< 1/2$ are open, we are done). Thus $y \in B_d(x, \delta)$. So the two open balls are equal.

Thus every open ball in each topology is an open ball in the other topology. The bases are equal, so the topologies are the same.

Problem 21.2 Suppose that $f : X \to Y$ is an isometric embedding (with an "e").

First, note that if $f(x) = f(x')$, then $d_X(x, x') = d_Y(f(x), f(x')) = 0$, so that $x = x'$. Thus f is injective. Let $x \in X$ and let $B_Y(f(x), \epsilon)$ be a basis element which contains $f(x)$. Claim: $f(B(x, \epsilon)) \subset B(f(x), \epsilon)$. For if $x' \in B(x, \epsilon)$, then $d_Y(f(x), f(x')) = d_X(x, x') < \epsilon$. Thus $f(B(x, \epsilon)) \subset B(f(x), \epsilon)$. So f is continuous. If we take a basis element for $f(X)$ in the subspace topology, it looks like $B = B_Y(f(x), \epsilon) \cap f(X)$. The image of this under the map f^{-1} is just $B_X(x, \epsilon)$, which is an open set in X.

21.3a

Let X_k be a metric space for $k = 1, \ldots, n$ with d_k the respective metric. Show that $\rho(x, y) = \max_k d_k(x, y)$ is a metric which induces the product topology on $X_1 \times \ldots \times X_n$.

Let $X = X_1 \times \ldots \times X_n$. Suppose that $x, x' \in X$. Then $\rho(x, x') = \max_k d_k(x_k, x'_k) \geq 0$, as each of the $d_k(x_k, x'_k) \geq 0$. If $x = x'$ then $x_k = x'_k$ for all k, and hence $\rho(x, x') = 0$. If $\rho(x, x') = 0$, then $x_k = x'_k$ for all k and hence $x = x'$. The symmetry follows from the fact that $d_k(x_k, x'_k) = d_k(x'_k, x_k)$ for all k. Let $x, y, z \in X$. For each k, $d_k(x, y) \leq d_k(x, z) + d_k(z, y)$. Note that $d_k(x, z) \leq \rho(x, z)$ and $d_k(z, y) \leq \rho(z, y)$ for all k. Thus we can replace the $d_k(x, z)$ and $d_k(z, y)$ on the right with $\rho(x, z)$ and $\rho(z, y)$ to obtain $d_k(x, y) \leq \rho(x, z) + \rho(z, y)$. But taking the maximum over all k on the left we obtain $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$. Thus ρ is a metric.

We show that the topology induced by ρ is the product topology on $X = X_1 \times \ldots \times X_n$. First, we show that the open balls relative to ρ are open sets in the product topology. Let $x \in X$. Then

$$B_\rho(x, \epsilon) = \{y \in X : \rho(x, y) < \epsilon\} = B_{d_1}(x_1, \epsilon) \times \ldots \times B_{d_n}(x_n, \epsilon);$$

for if $\rho(x, y) < \epsilon$, we must have $d_k(x_k, y_k) < \epsilon$ for all k, and conversely.

Now suppose that $U \subset X$ is open and $x \in U$. We produce an open ball $B = B_\rho(x, \epsilon)$ relative to ρ such that $x \in B \subset U$.

We can take a basic open set $B_{d_1}(x_1, \epsilon_1) \times \ldots \times B_{d_n}(x_n, \epsilon_n)$ containing x in the product topology (recall that the collection of Cartesian products of basis elements is a basis in the product topology) which is itself contained in U. Let $\epsilon = \min_k \epsilon_k$. Then

$$x \in B_\rho(x, \epsilon) \subset B_{d_1}(x_1, \epsilon_1) \times \ldots \times B_{d_n}(x_n, \epsilon_n) \subset U.$$

Thus U is open in the metric topology.

Problem 21.12

(a) Let $x \times y \in \mathbb{R}^2$ and suppose that $C = x + y$. Let $V = (C - \epsilon, C + \epsilon)$ be a basic nbhd of $x + y$. We produce a nbhd U of $x \times y$ in $\mathbb{R}^2$ such that $+(U) \subset V$, where $+(U)$ denotes the set of all sums of the ordered pairs in U. Let $U = (x - \frac{\epsilon}{2}, x + \frac{\epsilon}{2}) \times (y - \frac{\epsilon}{2}, y + \frac{\epsilon}{2})$. This is a nbhd of $x \times y$ in $\mathbb{R}^2$. If $w \times z \in U$, then

$$|w + z - C| = |w + z - (x + y)| = |(w - x) + (z - y)| \leq |w - x| + |z - y| < \epsilon.$$

Thus $+(U) \subset V$, and $+$ is continuous.

(b) Let $x, y \in \mathbb{R}$ and set $C = xy$. Let $V = (C - \epsilon, C + \epsilon)$ be a basic open nbhd of C. We construct a nbhd U of $x \times y$ in $\mathbb{R}^2$ such that $*(U) \subset V$, where $*(U)$ is all the products of the ordered pairs in U.

Pick $\delta < 1$ such that $\delta|y| < \epsilon/3$, such that $\delta|x| < \epsilon/3$ and $\delta^2 < \epsilon/3$. (Making $\delta = 1/N$ for large enough N suffices).

Note that if $\rho((w, z), (x, y)) < \delta$, then

$$|wz - xy| = |wz - wy + wy - xy| = |w||z - y| + |w - x||y| = |x||z - y| + |w - x||z - y| + |w - x||y| < \epsilon.$$

(c) There are some cases. Let $e : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ be the inversion map. Note that $e(A) = e^{-1}(A)$ for any subset of the domain.

(a) If $a < 0 < b$, then $e(a, b) = (-\infty, 1/a) \cup (1/b, \infty)$.

(b) If $0 < a < b$ or $a < b < 0$, then $e(a, b) = (1/b, 1/a)$.

(c) $e(0, a) = (0, 1/a)$ and $e(a, 0) = (1/a, 0)$.

(d) Subtraction: The map $H : 2\mathbb{R}^2 \to \mathbb{R}^2$ given by $(x, y) \to (x, -y)$ is continuous as its composition with each projection map is continuous. (Just a projection map or its negative, in each case). The composition $+ \circ H$ is the subtraction map, hence subtraction is continuous.

Quotient: The map $G : \mathbb{R} \times \mathbb{R} - \{0\} \to \mathbb{R} \times \mathbb{R} - \{0\}$ given by $G(x, y) = (x, 1/y)$ is continuous as composition with each projection map is continuous. Thus $* \circ G$, the quotient map, is continuous as well.