Math 54 - HW Solns 6

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23.1, 23.2, 23.3, 23.5, (23.7), 23.9, 23.10, 24.1, (24.2), 24.3, 24.7ab, 24.10

(We will sometimes say that a topology τ is connected as shorthand for saying that (X, τ) is connected).

Problem 23.1

As the discrete topology on $\mathbb{N}$ (which is disconnected) contains the indiscrete topology on $\mathbb{N}$ (which is connected), there is nothing to infer from $\tau \subset \tau'$ where τ is connected.

Let τ, τ' be topologies on X. If $\tau \subset \tau'$ and τ' is connected, then τ is connected. For suppose that $X = U \cup V$ for $U, V \in \tau$ disjoint. Then $U, V \in \tau'$, and as τ' is connected, $X = U \cup V$, and U and V are disjoint, we have that one of U or V is empty.

Problem 23.2

Let $B_n = A_1 \cup \ldots \cup A_n$. Then the family $\{B_n\}_{n=1}^{\infty}$ has intersection containing A_1. Also, $\cup_n B_n$ equals $\cup_n A_n$. Claim: the B_n are all connected. This proves the result by the theorem on unions of connected subspaces.

As A_1 is connected, the claim holds for $n = 1$. Suppose that B_k is connected. Then

$$B_{k+1} = A_1 \cup \ldots \cup A_k \cup A_{k+1} = B_k \cup A_{k+1}.$$

As B_k and A_{k+1} are connected and $B_k \cap A_{k+1} \supset A_k \cap A_{k+1} \neq \emptyset$, the union $B_k \cup A_{k+1} = B_{k+1}$ is connected.

Problem 23.3

Let $\{A_i\}$ be a collection of connected subspaces of X, and let A be a connected subspace of X such that $A \cap A_i \neq \emptyset$ for all i.

Let $B_i = A \cup A_i$ and note that $A \cup (\cup_i A_i) = \cup_i (A \cup A_i) = \cup_i B_i$. The subspaces B_i are connected as $B_i = A \cup A_i$, where the sets of the union are connected and they have non-empty intersection. As $B_i \cap B_j \supset A \neq \emptyset$, the union $\cup_i B_i$ is connected by the theorem on unions of connected subspaces.

Problem 23.5

Let X have the discrete topology. Suppose that $Y \subset X$ is connected. Note that Y has the discrete topology, as $\{y\} = Y \cap \{y\}$ for each $y \in Y$. If $y \in Y$, then $Y = \{y\} \cup Y - \{y\}$, which is a decomposition of Y into disjoint open subsets of Y. Thus one of these is empty, which must be $Y - \{y\}$. So $Y = \{y\}$.

The rationals $\mathbb{Q}$ are a totally disconnected space. Let $Y \subset \mathbb{Q}$ be a connected space and suppose that $y < z$ are distinct points of Y. We can take an irrational number p in (y, z). The decomposition $Y = (Y \cap (-\infty, p)) \cup (Y \cap (p, \infty))$ into disjoint non-empty open sets contradicts connectedness of Y.

Problem 23.9

Let $w \in X - A$ and $z \in Y - B$. Let $C_w = \{x \times y \in X \times Y : x = w \text{ or } y \notin B\}$ and $D_z = \{x \times y \in X \times Y : x \notin A \text{ or } y = z\}$. Note that $C_w = \cup_{y \notin B}(\{w\} \times Y \cup X \times \{y\})$. This is a union of connected spaces as $\{w\} \times Y$ and $X \times \{y\}$ are connected and they both contain (w, y). The intersection $\cap_{y \notin B}(\{w\} \times Y \cup X \times \{y\})$ contains $\{w\} \times Y$, so C_w is connected. Similarly, D_z is connected. They both contain $w \times z$, so their union $C_w \cup D_z$ is connected. Claim: $C_w \cup D_z = X \times Y - A \times B$. We prove the right-left inclusion first. As in HW 3, $X \times Y = (X - A) \times Y \cup X \times (Y - B)$, and $(X - A) \times Y \subset D_z$ and $X \times (Y - B) \subset C_w$. The other inclusion is similar:

$$C_w = \{w\} \times Y \cup X \times (Y - B) \subset (X - A) \times Y \cup X \times (Y - B) = X \times Y - A \times B.$$

Thus $C_w \subset X \times Y - A \times B$. Similarly $D_z \subset X \times Y - A \times B$. Thus $X \times Y - A \times B$ is connected.

Problem 23.10

(a) Assume that K is non-empty, for otherwise $X_K = \{\mathbf{a}\}$, which is a connected space. Note that $X_K = \prod_{j \in J} U_j$, where $U_j = X_j$ if $j \in K$ and $U_j = \{a_j\}$ if $j \notin K$. We show that X_K is homeomorphic to $Z_K = \prod_{j \in K} X_j$, then invoke the connectedness of the finite product $\prod_{j \in K} X_j$ of connected spaces. Define $\phi : X_K \to Z_K$ given by projection, i.e. $\phi(s_k = s_k$ for $s \in X_K$. This is continuous as projection maps are continuous. Define $\psi : Z_K \to X_K$ by

$$\psi(s)_j = \begin{cases} s_j & j \in K \\ a_j & j \notin K \end{cases}$$

The composition $\pi_j \circ \psi$ with any of the projection maps π_j for $j \in J$ yields a continuous map. Thus ψ is continuous. As the maps ϕ and ψ are inverses of one another, and each is continuous, they are homeomorphisms.

(b) Each of the spaces X_K contains $\mathbf{a}$, and as each X_K is connected, their union is connected.

(c) Suppose that $\prod_j U_j$ is a basis element of the product topology. We show that it contains a point of Y, which proves the claim, for then any nbhd of any point of X contains a point of Y. By the definition of the product topology, there is a finite subset $K \subset J$ such that $U_j = X_j$ for all $j \notin K$. We construct a point of Y as follows: for each $j \notin K$, let $s_j = a_j$, and for each $j \in K$, let s_j be an arbitrary point of U_j. Then $s \in \prod_j U_j$.

Problem 24.1

(a) Note that $(0, 1)$ is not compact as the open cover $\{(1/n, 1 - (1/n))\}_{n \in \mathbb{N}}$ contains no finite subcover. Similarly $(0, 1]$ has the open cover $\{(1/n, 1]\}_{n \in \mathbb{N}}$ of $(0, 1]$ has no open subcover. As compactness is preserved under homeomorphism, neither of these spaces are homeomorphic to the compact space $[0,1]$. Suppose $f : (0,1] \to (0,1)$ is a homeomorphism. Then $g = f|_{(0,1)} : (0,1) \to (0,1) - \{f(1)\}$ is a bijection. As it is a restriction of a continuous function g is continuous. The inverse of g is the restriction of f^{-1} to $(0,1) - \{f(1)\}$, so it is also continuous. Thus g is a homeomorphism. But $(0,1) - \{f(1)\} = (0, f(1)) \cup (f(1), 1)$ is a disconnected space. But g must preserve connectedness and its domain $(0,1)$ is connected. This contradiction shows that no such homeomorphism exists.

(b) Let $f : [-1,1] \to (-1,1)$ be given by $f(x) = x/2$, and let $g : (-1,1) \to [-1,1]$ be the inclusion map. Both are continuous bijections. The map $f^{-1} : [-1/2, 1/2] \to [-1,1]$ given by $f^{-1}(y) = 2y$ is continuous, as is the identity map $g^{-1} : (-1,1) \to (-1,1)$. Thus both f and g are embeddings. The spaces $[-1,1]$ and $(-1,1)$ are not homeomorphic.

(c) Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a homeomorphism for some $n > 1$. Then $f|_{\mathbb{R}^n - \{0\}} : \mathbb{R}^n - \{0\} \to \mathbb{R} - \{f(0)\} = (-\infty, f(0)) \cup (f(0), \infty)$ is also a homeomorphism. But we showed in class that the domain of this restricted function is path-connected, hence also connected. This contradicts that the continuous image of a connected space is connected.

Problem 24.3 Let $g(x) : [0,1] \to \mathbb{R}$ be given by $g(x) = f(x) - x$. Note that g is continuous as subtraction is a continuous operation. Then $g(0) \leq 0$ and $g(1) \geq 0$. If either of these inequalities are equalities, we're done. Else $g(0) < 0 < g(1)$, so that there exists $c \in [0,1]$ such that $g(c) = 0$. Then we're done.
For the spaces $[0,1)$ and $(0,1)$ the maps $f(x) = \frac{x+1}{2}$ are both continuous. If $f(x) = x$, then $x = 1$, a contradiction. Thus f has no fixed points on the domains $[0,1)$ or $(0,1)$
Problem 24.7

(a) Let $x \neq x'$ in X, say $x < x'$. Then $f(x) < f(x')$. So f is injective. Let (w,z) be a basis element in Y. We show that $f^{-1}(w,z)$ is open (the cases for basis elements which contain the largest/smallest elements are much the same). Let $a, b \in X$ such that $f(a) = w$ and $f(b) = z$. As f is order-preserving, $a < b$. Then $(a,b) = f^{-1}(w,z)$, as we prove. Let $x \in (a,b)$. Then $a < x < b$, so that $w = f(a) < f(x) < f(b) = z$. Thus $(a,b) \subset f^{-1}(w,z)$. Suppose that $x \in f^{-1}(w,z)$. Then $f(a) = w < f(x) < z = f(b)$. As f preserves order, we must have $a < x < b$. Thus $x \in (a,b)$. Thus $(a,b) \subset f^{-1}(w,z)$, and the pre-image of (w,z) is (a,b). Notice that f is injective and surjective and order-preserving. Thus the inverse function is also order-preserving and hence continuous. Then f is a homeomorphism.

(b) Suppose that $x < y$ in $[0,\infty)$. Then

$$f(y) - f(x) = y^n - x^n = (y-x)(y^{n-1} + y^{n-2}x + \ldots + yx^{n-2} + x^{n-1})$$

from algebra. But this is the product of two positive numbers, hence positive. Thus $f(y) > f(x)$. So f preserves order. Suppose that $x \in [0,\infty)$. If $x = 0$, then $x = f(0)$, so that x belongs to the image of f. Suppose that $0 < x$. Then $(x+1)^n =$

$x^n + nx^{n-1} + \ldots + nx + 1 > x$. Thus $f(0) < x < f(x+1)$. The intermediate value theorem gives $c \in [0, x+1]$ such that $f(c) = x$.

Problem 23.10

(This exercise generalizes to any $\mathbb{R}^n$ for $n > 1$, and the statement is trivially true for $n = 1$). Let $x_0 \in U$ and let $V = \{y \in U : \text{ there exists a path in } U \text{from } x_0 \text{ to } y\}$. Claim: V is an open subset of U. For let $y \in V$. As U is an open subset of $\mathbb{R}^2$, we can select an open ball $B(y, \epsilon) \subset U$. Let $z \in B(y, \epsilon)$ and define $f : [0, 1] \to B(y, \epsilon)$ by $f(t) = (1-t)y + tz$. This is a path in U (contained in the open ball $B(y, \epsilon)$ in fact) from y to z. As $y \in V$, there exists a path g in U from x to y. Then the map $h : [0, 1] \to U$ given by

$$h(t) = \begin{cases} g(2t) & t \in [0, \frac{1}{2}] \\ f(1-2t) & t \in [\frac{1}{2}, 1] \end{cases}$$

is a path from x_0 to z. Thus $z \in V$, and so every point in V has a nbhd which is contained in V. Thus V is open.

Let W be the set of all points $w \in U$ to which x_0 is not connected by a path in U. For a given $x \in U$ let V_x be the set of all points in U which are connected to x by a path in U. Then $U = V \cup (\cup_{w \in W} V_w)$. Note that $V \cap V_w = \emptyset$, or else there is a point z which is connected by a path in U to x_0 and connected by a path in U to a point which is not path-connected to x_0. This creates a contradiction, for we can paste the paths together as in the first part of the problem. Thus U is written as the disjoint union of open sets $V \cup (\cup_{w \in W} V_w)$. Since U is connected and V contains x_0 via the constant path $f(t) = x_0$, we must have $\cup_{w \in W} V_w = \emptyset$. But $W \subset \cup_{w \in W} V_w$ as $w \in V_w$ for all w. Thus $W = \emptyset$, and every point in U is joined to x_0 by a path. As we can paste paths together, any two points in U are joined by a path.