Math 54 - HW Solns 7

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26.1, 26.5, 26.7, 26.8, 27.6

Problem 26.1

(a) Suppose that $\tau' \supset \tau$ and X is compact when equipped with τ'. Then X is compact when equipped with τ. For if $\{U_i\} \subset \tau$ is an open cover of X, we have $\{U_i\} \subset \tau'$ as well. As X is compact when equipped with τ', we can extract a finite subcover $\{U_{i_1}, \ldots, U_{i_n}\}$. This is also a finite subcover in τ, as each of the sets $U_{i_k} \in \tau$. Thus every cover of X with sets of τ has a finite subcover, so that X is compact when equipped with τ.

(b) Suppose that $\tau \subset \tau'$, where both topologies are compact Hausdorff. We show that $\tau = \tau'$. Let X denote the set X equipped with τ, X' the same set equipped with τ'. By problem 18.3 and $\tau \subset \tau'$, the identity function $i_X : X' \to X$ is continuous. As X' is compact and X is Hausdorff, the map is closed as well. Then the inverse map $j_X : X \to X'$ is continuous ($j_X(x) = x$)), for if $C \subset X'$ is closed, $j_X^{-1}(C) = i_X(C)$ is closed. Thus the function i_X and its inverse are both continuous, so that $i_X : X' \to X$ is a homeomorphism. By problem 18.3 this implies that $\tau = \tau'$.

Problem 26.5

We apply lemma 26.4 of Munkres, which arose in the proof that compact subsets of Hausdorff spaces are closed. For each $x \in A$, pick open sets U_x and V_x such that $x \in U_x$ and $B \subset V_x$. Then the collection $\{U_x\}_{x \in A}$ is an open cover of A, so that we can select a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$ so that $A \subset U = U_{x_1} \cup \ldots \cup U_{x_n}$. Then the open set $V = V_{x_1} \cap \ldots \cap V_{x_n}$ contains B, as each of the sets V_{x_k} does. The sets U and V are disjoint because if $y \in V$, then $y \in V_{x_k}$ for all k. But $U_{x_k} \cap V_{x_k} = \emptyset$, so this implies that $y \notin U_{x_k}$. As this is true for all $y \notin U$. Thus the open sets U and V are disjoint nbhds of the sets A and B respectively. (We will use this to prove that all compact Hausdorff spaces are *normal* next week).

Problem 26.7

Let $C \subset X \times Y$ be closed. We show that the complement $X - \pi_1(C)$ is open. If this latter set is empty, we're done. So let $x \notin \pi_1(C)$. We construct a nbhd of x which is included in $X - \pi_1(C)$. Then $\{x\} \times Y \subset X \times Y - C$, as for any $y \in Y$ the image $\pi_1(x \times y) = x \notin \pi_1(C)$. As the second factor Y is compact, we can apply the tube lemma to the slice $\{x\} \times Y$ and the open set $X \times Y - C$ containing the slice to get a nbhd W of x such that $W \times Y \subset X \times Y - C$. Then $W \subset X - \pi_1(C)$, as given any $x \times y \in C$, we have $x \notin W$ and so $\pi_1(C)$ does not intersect W. Thus around any point of $X - \pi_1(C)$ we have constructed a nbhd which is contained in $X - \pi_1(C)$.

Problem 26.8

Suppose that f is continuous. We show that the graph is closed by showing that its complement is open. Let $x \times y \notin G_f$, so that $f(x) \neq y$. We construct a nbhd of $x \times y$ in the product space $X \times Y$ which does not intersect G_f. As $f(x) \neq y$ in the Hausdorff space Y, we can take disjoint nbhds V_1 and V_2 of $f(x)$ and y respectively. Then $f^{-1}(V_1)$ is a nbhd of x, and so the product $W = f^{-1}(V_1) \times V_2$ is a nbhd of $x \times y$ in $X \times Y$. Claim: W does not intersect G_f. For if $p \times q \in W$, then $f(p) \in V_1$ by $p \in f^{-1}(V_1)$ and $q \in V + 2$ by construction. But $V_1 \cap V_2 = \emptyset$, so $q \neq f(p)$. Thus $p \times q \notin G_f$, and so W contains no points of G_f. Thus we have constructed the required nbhd.

Now suppose that G_f is closed in $X \times Y$. Let $x_0 \in X$ and let V be a nbhd of $f(x_0)$. We construct a nbhd U of x_0 such that $f(U) \subset V$. If $f(X) \subset V$, then we are done, for we can set $U = X$. Thus assume that $f(X) \not\subset V$. The set $X \times (Y - V)$ is closed in $X \times Y$ as it is a product of closed sets. Thus $G_f \cap (X \times (Y - V))$ is closed. Thus $C = \pi_1(G_f \cap (X \times (Y - V)))$ is a closed subset of X, by the previous problem. Suppose that $x \in U = X - C$. The point $x \times f(x)$ is in G_f. If $f(x) \in Y - V$, then $x \times f(x) \in X \times (Y - V)$, so that x would lie in the image C. Thus $f(x) \in V$. Thus $f(U) \subset V$. Also note that $x_0 \notin C$, for otherwise $x_0 \times f(x_0) \in X \times (Y - V)$, which contradicts $f(x_0) \in V$.

Problem 27.6 (The Cantor Set)

(a) Let Y be a subspace of C with more than one point. We show that Y is disconnected. Suppose that $x, y \in Y$. Let $d(x, y) = \epsilon$, and pick n such that $\frac{1}{3^n} < \epsilon$ (we can do this by the note on density from last week). Consider the greatest integer p such that $\frac{p}{3^n} < x$ and the least integer q such that $y < \frac{q}{3^n}$. Then $q - p > 1$, as $\frac{1}{3^n} < x - y < \frac{q-p}{3^n}$.

    Then if x and y are contained in the same interval $[\frac{p}{3^{n-1}}, \frac{p+1}{3^{n-1}}]$, for some integer p, we must have that $x \in [\frac{p}{3^{n-1}}, \frac{3p+1}{3^n}]$ and $y \in [\frac{3p+2}{3^n}, \frac{p+1}{3^{n-1}}]$. For otherwise the distance between x and y is at most $\frac{1}{3^n}$, contradicting the above inequality.

    Pick a number $z \in (\frac{3p+1}{3^n}, \frac{3p+2}{3^n})$. Then $z \notin C$, and the open sets $(-\infty, z) \cap Y$ and $(z, \infty) \cap Y$ form a separation of Y.

(b) Each set A_n is closed, for A_0 is closed, and the complement of an open set in a closed set is closed. Thus $C = \cap A_n$ is closed. Clearly C is bounded, as A_0 is bounded. Thus C is closed and bounded, hence compact by the Heine-Borel theorem.

(c) We prove via induction that:

    (a) A_m is a union $[0, \frac{1}{3^m}] \cup \ldots [1 - \frac{1}{3^m}, 1]$ of disjoint closed intervals, where the intervals have the form $[\frac{j}{3^m}, \frac{j+1}{3^m}]$

    (b) The endpoints of the closed intervals in each A_n are contained in each A_m.

    (c) If $[a, b]$ and $[c, d]$ are two disjoint intervals in A_m, with $b < c$, then $\frac{1}{3^m} \leq c - b$.

The claim is true for $m = 0$, as $[0,1]$. Suppose that the claim is true for m, so that $A_m = [0, \frac{1}{3^m} \cup \ldots \cup [1 - \frac{1}{3^m}, 1]$. Then

$$A_{m+1} = A_m - \cup_{k=0}^{\infty} \left(\frac{1+3k}{3^{m+1}}, \frac{2+3k}{3^{m+1}} \right).$$

Each of the sets $\left(\frac{1+3k}{3^{m+1}}, \frac{2+3k}{3^{m+1}}\right)$ intersects most one of the intervals $[a,b]$ in A_m. Otherwise, assuming that $z \in \left(\frac{1+3k}{3^{m+1}}, \frac{2+3k}{3^{m+1}}\right) \cap [a,b]$ and $z' \in \left(\frac{1+3k}{3^{m+1}}, \frac{2+3k}{3^{m+1}}\right) \cap [c,d]$ for $[a,b]$, $[c,d]$ closed intervals from A_m with $b < c$, we obtain $b - c < z' - z < \frac{1}{3^{m+1}}$. But the intervals are separated by at least $\frac{1}{3^m}$ by (3) of the induction hypothesis, so $b - c \geq \frac{1}{3^m}$.

An interval in A_m has the form $[\frac{j}{3^m}, \frac{j+1}{3^m}]$. Thus if $\left(\frac{1+3k}{3^m}, \frac{2+3k}{3^m}\right)$ intersects some interval $[\frac{j}{3^m}, \frac{j+1}{3^m}]$, it must be contained within it. On removing the open interval, we do not disturb the endpoints of the closed interval. Thus the endpoints are contained in A_{m+1}.

Each of the closed intervals in A_m is split into two disjoint closed intervals in this way. Thus we obtain a finite disjoint union of closed intervals equaling A_{m+1}.

If we take two such intervals in A_{m+1}, they are separated by at least $\frac{1}{3^{m+1}}$. For if they came from the same interval in A_m, we have removed an interval of length $\frac{1}{3^{m+1}}$ from that interval to separate them. If they came from different intervals in A_m, they are separated by at least $\frac{1}{3^m}$.

(d) Suppose that $x \in C$, and let $\epsilon > 0$. We show that the ball of radius ϵ about x contains another point of C. Pick n such that $\frac{1}{3^n} < \epsilon$. As $x \in C$, we have that $x \in A_n$. Then pick a closed interval $[a,b] \subset A_n$ of length $\frac{1}{3^n}$ in A_n such that $x \in [a,b]$. The endpoint $a \in C$ by part c. Then $d(x,a) = |x - a| \leq |a - b| = \frac{1}{3^n} < \epsilon$. Thus the ball of radius ϵ contains a point of C different from x.

(e) By the theorem 27.7 of Munkres, we only have to show that C is compact, Hausdorff, and without isolated points. But we have shown C to be compact and without isolated points, and any subspace of a Hausdorff space is Hausdorff. As C is non-empty (it contains, e.g., 1), it must be uncountable.