Math 54 - HW Solns 8

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August 16, 2012

Problem 30.3

Let A be an uncountable subspace of the second-countable space X. Then A is second countable, as subspaces of second countable spaces are second countable. Thus A has a countable dense subset B. Then $A = \overline{B} = B \cup B'$. Then B' must be uncountable, or else A would be uncountable. Every element of B' is a limit point of B and hence of A also. Thus A has uncountably many limit points.

Problem 30.9 Let X be a Lindelof space, let A be a closed subspace. We show that A is Lindelof also. Let $\{V_i\}_{i \in I}$ be an open cover of A in the subspace topology. We can choose for each i and open subset $U_i \subset X$ such that $U_i \cap A = V_i$. Then $\cup U_i$ contains A. Thus the collection $\{U_i\} \cup \{X - A\}$ is an open cover of X, as A is closed. Take a countable subcover $\{U_{i_n}\}_{n=1}^{\infty} \cup \{X - A\}$. Discard the set $X - A$ and consider the collection $\{V_{i_n}\}_{n=1}^{\infty}$. This is a subcollection of $\{V_i\}$. Claim: it is a subcover. For if $a \in A$, then some U_{i_n} or $X - A$ contains a, as we chose a subcover of X from $\{X - A\} \cup \{U_i\}_{i \in I}$. As $X - A \cap A = \emptyset$, it must be the case that $a \in U_{i_n}$ for some n. Then $a \in A \cap U_{i_n} = V_{i_n}$, so that $\{V_{i_n}\}_{n=1}^{\infty}$ is a countable subcover.

Problem 30.12 Suppose that X is first countable. Suppose that $y \in f(X)$, so that $y = f(x)$ for some $x \in X$. We can take a countable basis $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ for X at x. Claim: $f(\mathcal{B}) = \{f(B_n)\}_{n=1}^{\infty}$ is a countable basis at y. This is countable because it is the image of a countable set. As f is open, it is a collection of open sets in the subspace topology, because $f(B_n) \cap f(X) = f(B_n)$, and as $x \in B_n$ for all n, we have $f(x) \in f(B_n)$ for all n. Thus all the sets $f(B_n)$ are nbhds of $f(x) = y$ in the subspace topology. Now let V be a nbhd of y in $f(X)$, so that $V = W \cap f(X)$ for some open set $W \subset Y$. We produce n such that $f(B_n) \subset V$. Consider the preimage $U = f^{-1}(W) \subset X$. Then $x \in U$, as $f(x) = y \in V$. As f is continuous, U is open and therefore a nbhd of x. Thus we can take n such that $B_n \subset U$. Then $f(B_n) \subset f(U) \subset V$, proving the claim. Thus $f(\mathcal{B})$ is a basis at y.

Suppose that X is second-countable, with $\mathcal{B} = \{B_n\}_{n=1}^{\infty}$ a basis for the topology on X. We show that $f(X)$ is second-countable in the subspace topology with countable basis $\{f(B_n)\}_{n=1}^{\infty}$. Note that as above each of these sets is open in the subspace topology. Let $V \subset f(X)$ be open in the subspace topology, so that there exists an open set $W \subset Y$ such that $W \cap f(X) = V$. Consider a point $y \in V$. We find an element $f(B_n)$ containing y and contained within V. Note that $U = f^{-1}(W)$ is open in X, and $x \in U$ as $f(x) \in V \subset W$. Thus we can take B_n such that $x \in B_n \subset U$. Then $y = f(x) \in f(B_n) \subset f(U) = V$. Thus about any point of the open set V, we can take $f(B_n)$ contained in V. Thus $f(\mathcal{B})$ is a basis for the topology on $f(X)$.

Problem 31.1

Suppose that X is regular and x and y are distinct points. We construct nbhds U and V of x and y respectively such that $\overline{U} \cap \overline{V} = \emptyset$. Note that as X is regular, X is Hausdorff. Thus there exist disjoint nbhds P and Q of x and y. Then $x \in P$, where P is open. As X is a regular space, we can take a nbhd U of x such that $\overline{U} \subset P$. Similarly we can take a nbhd V of y such that $\overline{V} \subset Q$. Then $\overline{U} \cap \overline{V} \subset P \cap Q = \emptyset$, so that U and V are the required nbhds.

Problem 31.3

Claim: Let X be an ordered set in the order topology and let (a, b) be an interval in X. Then $\overline{(a, b)} \subset [a, b]$. For if $x < a$, then $(-\infty, a)$ is a nbhd of x which does not meet (a, b), and if $x > a$, then (b, ∞) is the same. The same logic applies to show that $\overline{[a, b)} \subset [a, b]$ and $\overline{(a, b]} \subset [a, b]$.

Proof of regularity We show that X is regular using the formulation of the lemma. Suppose that $x \in U$, where U is an open subset of X. We construct a nbhd V of x such that $\overline{V} \subset U$. As U is open and $x \in U$, there is some interval I containing x such that $I \subset U$, by the definition of the basis for the order topology.

Now, suppose that $I = [x, b)$. If $(x, b) = \emptyset$, then $V = [x, b) = \{x\}$ satisfies $V = \overline{V} \subset U$, as one-point sets are closed in any order topology. If $a \in (x, b)$, consider the set $V = [x, a)$. Then $\overline{V} \subset [x, a] \subset [x, b) \subset U$, and we are done.

The proof when $I = (a, x]$ is similar.

Suppose that $I = (a, b)$. If $(a, x) = \emptyset = (x, b)$, then $(a, b) = \{x\}$ is a nbhd of x equaling its closure as above and contained in U.

If (a, x) is empty and $c \in (x, b)$, then $(a, c) = [x, c)$ is a nbhd of x with closure contained in $[x, c] \subset (a, b) \subset I \subset U$.

If (a, x) and (x, b) are both non-empty, with $c \in (a, x)$ and $d \in (x, b)$, then (c, d) is a nbhd of x with closure $\overline{(c, d)} \subset [c, d] \subset (a, b) \subset U$.

Problem 31.5 Let $f, g : X \to Y$ be continuous with Y Hausdorff. Note that $\Delta = \{y \times y : y \in Y\}$ is closed in $Y \times Y$, as Y is Hausdorff. The map $f \times g : X \to Y \times Y$ given by $(f \times g)(x) = f(x) \times g(x)$ is continuous, as each of the coordinate functions f, g is. The set $\{x : f(x) = g(x)\}$ is simply the preimage $(f \times g)^{-1}(\Delta)$, necessarily closed as $f \times g$ is continuous and Δ is closed.