Math 54 - HW Solns 9

Dan Crytser

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Problem 32.1

Let X be a normal space and let Y be a closed subspace. Suppose that A, B are disjoint closed subsets of Y. As Y is closed, this implies that A and B are closed subsets of X, because they must equal the intersection of the closed set Y with some closed subsets of X. In X there are disjoint nbhds U and V of A and B respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y which contain A and B respectively. Thus Y is normal.

Problem 32.4

We show that a regular Lindelof space is normal. Let X be a regular Lindelof space, and suppose that A and B are disjoint closed subsets of X. For each $a \in A$, we know that a belongs to the open set $X - B$. Thus by regularity we can select a nbhd of a, say U_a, such that $\overline{U_a} \subset X - B$. By a previous assignment, we know that A is Lindelof. So from the open cover $\{U_a \cap A\}$ of A, we can extract a finite subcover $\{U_{a_n} \cap A\}_{n=1}^{\infty}$, where each U_{a_n} has closure contained in $X - B$. Re-index U_{a_n} simply as U_n, and consider the collection $\{U_n\}_{n=1}^{\infty}$ of open subsets of X. Notice that $A \subset \cup_{n=1}^{\infty} U_n$ and $\overline{U_n} \subset X - B$ for all n. We can similarly construct a collection $\{V_n\}$ of open subsets of X such that $B \subset \cup_{n=1}^{\infty} V_n$ and $\overline{V_n} \subset X - A$ for all n.

As in the proof that second-countable spaces are regular, we define two new sequences of open sets:

$$U_n' = U_n - \cup_{i=1}^{n} \overline{V_i}$$

and

$$V_n' = V_n - \cup_{i=1}^{n} \overline{U_i}.$$

Then the argument given in class shows that $U_n' \cap V_m' = \emptyset$ for all n, m, and that $U = \cup_{n=1}^{\infty} U_n'$ and $V = \cup_{n=1}^{\infty} V_n'$ are the desired open sets.

Note: we have shown that all regular Lindelof spaces are normal. It is not the case that all regular separable spaces are normal. To see this, it suffices to show that $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is regular, as the product of regular spaces, and separable, as the (countable) product of separable spaces, but not normal. Thus, in some sense, separability is weaker than Lindelofness.

Problem 32.6 We abbreviate the condition on separated sets to (*). Suppose that X is completely normal, so that every subspace of X is normal. We show that it satisfies (*).

Suppose that A and B are separated, so that $A\cap\overline{B}=\emptyset=\overline{A}\cap B$. Consider the subspace $Y=X-(\overline{A}\cap\overline{B})=(X-\overline{A})\subset(X-\overline{B})$ (using deMorgan's laws). Note that $A,B\subset Y$, as $A\subset X-\overline{B}$ and $B\subset X-\overline{A}$ by the separation condition.

Let C be the closure of A in the subspace Y, and let D be the closure of B in the subspace Y. We prove a statement that Munkres proves: the closure of A relative to Y is the intersection of $\overline{A}$ with Y.

Then $C=\overline{A}\cap Y$. For if $x\in C$, then $x\in Y$, as C is a subset of Y by definition. Also, if W is any nbhd of x in X, then $W\cap Y$ is a nbhd of x in Y, hence it contains an element of A. Thus W contains an element of A, and $x\in\overline{A}$. So $C\subset\overline{A}\cap Y$. Suppose that $x\in\overline{A}\cap Y$. If $Z=W\cap Y$ is a nbhd of x in Y, where W is an open set of X, then we must have a point $a\in W\cap A$, as $x\in\overline{A}$. Then $a\in Z$, as $A\subset Y$. Thus Z contains an element of A, and so $x\in C$. Thus $\overline{A}\cap Y=C$.

Similarly $\overline{B}\cap Y=D$. These are both closed subsets of the space Y. They are disjoint, for

$$C\cap D=\overline{A}\cap Y\cap\overline{B}\cap Y=(\overline{A}\cap\overline{B})\cap(X-(\overline{A}\cap\overline{B}))=\emptyset.$$

Thus, we can take disjoint nbhds of in Y, say U and V. As U is open in the open subspace Y, it is open in X as well. Similarly V is open in X. Thus U and V are disjoint nbhds of A and B, and X satisfies (*).

Suppose that X satisfies (*). We show that it is completely normal. Suppose that Y is a subspace.

Problem 33.2

(a) Let X be a connected normal space with distinct points a and b. Then $A=\{a\}$ and $B=\{b\}$ are disjoint closed subsets of X, as X is T_1. Thus there is a function $f:X\to[0,1]$ such that $f(a)=0$ and $f(b)=1$. We show that this function is surjective. Let $c\in(0,1)$. Then $f(a)<c<f(b)$, so that we can invoke the intermediate value theorem for the connected space X to produce $x\in X$ such that $f(x)=c$. Thus every point in $[0,1]$ is in the image of f. If $\phi:\mathbb{N}\to X$ were a surjection, then $f\circ\phi:\mathbb{N}\to[0,1]$ would be a surjection, implying that $[0,1]$ is countable. But $[0,1]$ is uncountable, as it is a compact Hausdorff space with no isolated points. So there is no surjection $\phi:\mathbb{N}\to X$, and so X is uncountable.

(b) Let X be a connected regular space with more than one point. We show that X is uncountable by means of a contradiction. In particular, we show that a countable regular space must be normal, so that a countable connected regular space with more than one point would provide a counterexample to the above. So, suppose that X is countable $X=\{x_1,\dots x_n\}$. This implies that X is Lindelof For if $\{U_i\}_{i\in I}$ is an open cover, we can for each n choose $i_n\in I$ such that $x_n\in U_{i_n}$. Then $\{U_{i_n}\}_{n=1}^{\infty}$ is a countable subcover.

Problem 33.3 Let A,B be disjoint closed subsets of the metric space X, and define

$$f(x)=\frac{d(x,A)}{d(x,A)+d(x,B)}.$$

First, we show that the function $f(x)$ so defined is continuous. If $x \in X$, then $d(x, A) = 0$ if and only if $x \in \overline{A} = A$. For if $\inf_{a \in A} d(x, a) = 0$, then for each n we can pick $a_n \in A$ such that $d(x, a_n) < \frac{1}{n}$. This sequence $(a_n) \subset A$ then converges to x, hence $x \in \overline{A} = A$. Similarly $d(x, B) = 0$ if and only if $x \in B$. Thus it is impossible for $d(x, A) + d(x, B)$ to equal 0, for otherwise we would have $x \in A \cap B = \emptyset$. Thus $g(x) = d(x, A) + d(x, B)$ is nonzero, and it is continuous by argument on page 175. Similarly, $h(x) = d(x, A)$ is continuous. Thus map $x \to (h(x), g(x))$ is continuous, and so on composing with the (continuous) division operation in $\mathbb{R}$, we obtain that $x \to \frac{d(x,A)}{d(x,A)+d(x,B)}$ is continuous.

Suppose that $x \in A$. Then $f(x) = 0$, as $d(x, A) = 0$. Suppose that $x \in B$. Then $d(x, B) = 0$, and $f(x) = \frac{d(x,A)}{d(x,A)} = 1$.

Problem 34.3 Let X be a compact Hausdorff space. Suppose that X is second countable. As X is compact Hausdorff, X is normal and therefore regular. Thus X is a regular and second countable, hence metrizable by the Urysohn metrization theorem.

Suppose that X is metrizable. Then X is second countable. As X is a compact metrizable space, it has a countable basis. Thus X is second countable.