MATH 54 - LECTURE 1: INTRO TO TOPOLOGY; COUNTABLE AND UNCOUNTABLE SETS

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1. Goals for Today's Lecture

(1) Students understand the basic idea of topology as a generalized study of continuity and limits.
(2) Students understand all the requirements of the course, viz. homework collection policy, use of x-hours, exams, grading distribution.
(3) Students understand the difference between countable and uncountable sets. Cantor diagonal argument.

2. Evaluation of Goals

3. The Idea of Topology

Definition We denote the real number line by $\mathbb{R}$. For a positive integer $n > 0$ we let $\mathbb{R}^n$ denote the set of all n-tuples $(x_1, \ldots, x_n)$ with each x_i a real number.

Example The triple $(1, 7, 0)$ is in the set $\mathbb{R}^3$, which we think of as three-dimensional space. The quadruple $(-1, 0, 0, 2)$ is in $\mathbb{R}^4$.

Definition Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then the *length* or *norm* of x is the non-negative number $||x|| = \sqrt{x_1^2 + \ldots + x_n^2}$.

Example If $n = 1$ then $||x|| = |x|$ the absolute value of the number x.

Example The length of $(1, 2, 0)$ is $\sqrt{5}$. **Example** The length of $(-1, 0, 7, 0)$ is $\sqrt{1 + 49} = 5\sqrt{2}$.

Definition A function $f : X \to Y$ is a rule that assigns to each $x \in X$ a unique $f(x) \in Y$.

Definition A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x \in \mathbb{R}$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if y is a real number and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

This definition seems a little bit strange at first, but if we draw a diagram, we can see that no matter how small a tolerance interval we inscribe about the number $f(x)$, we can force f to map into that tolerance interval by looking at a very small control interval around x.

It is easy to use the norm functions on the various $\mathbb{R}^n$ to generalize the definition of continuity to higher dimensions.

Definition A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at the point $x \in \mathbb{R}^n$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $y \in \mathbb{R}^n$ and $||x - y|| < \delta$, then $||f(x) - f(y)|| < \epsilon$. A function is continuous on $\mathbb{R}^n$ if it is continuous at every point of $\mathbb{R}^n$

Example Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(p, q) = p$, so that f merely picks off the x-coordinate of the point (p, q) in the plane. We claim that f is continuous at every point $x = (x_1, x_2)$ of the plane. How do we prove this: We let $\epsilon > 0$. We seek $\delta > 0$ so that if $y \in \mathbb{R}^n$ with $||x - y|| < \delta$, then $|x_1 - y_1| < \epsilon$. I claim that if we set $\delta = \epsilon$, the latter inequality holds. Suppose that $y \in \mathbb{R}^n$ with $||x - y|| < \epsilon$, which is to say $(x_1 - y_1)^2 + (x_2 - y_2)^2 < \epsilon^2$ on squaring both sides of the inequality.

$$(x_1 - y_1)^2 \leq (x_1 - y_1)^2 + (x_2 - y_2)^2 < \epsilon^2.$$

Now we can write $(x_1 - y_1)^2 < \epsilon^2$ and take square roots, so that $|x_1 - y_1| < \ epsilon$. So we have shown that if $||x - y|| < \epsilon$, then $|f(x) - f(y)| < \epsilon$. Thus f is continuous at x. As there is absolutely nothing special about x – nothing about its coordinates was specified – we see that f is continuous at every point of $\mathbb{R}^n$.

This definition of continuity is phrased in an extremely *local* fashion. First you pick a point, then you show that you can control these tolerance intervals around the point. One of the focuses of topology is in studying continuity in a less local, more *global* definition. To do this, we need to look more carefully at the structure of $\mathbb{R}^n$.

Definition If $x, y \in \mathbb{R}^n$, then $||x - y|| = d(x, y)$ is called the *distance* between x and y.

Definition Let $x \in \mathbb{R}^n$. Then for $\epsilon > 0$ we define $B(x, \epsilon)$ to be the set of all y within distance ϵ of x, i.e.

$$B(x, \epsilon) = \{y \in \mathbb{R}^n : ||x - y|| < \epsilon\}.$$

We call $B(x, \epsilon)$ the *open ball of radius* ϵ *about* x.

Example Let $0 = (0, 0)$ the origin in the plane $\mathbb{R}^2$. Then $B(0, 1)$ is the disk of radius 1, centered at the origin, without its boundary circle.

Definition Let $U \subset \mathbb{R}^n$ be a subset of $\mathbb{R}^n$. Then we say that U is *open* if U is empty or if U is nonempty and for every $x \in U$ we can find $\epsilon > 0$ so that $B(x, \epsilon) \subset U$. That is, for each x, everything within a suitably small distance of x is contained in U. (The distance ϵ typically varies with each x). If a subset of $\mathbb{R}^n$ is open we call it an open subset or open set.

Example The open ball $B(0, 1) \subset \mathbb{R}^2$ is open in $\mathbb{R}^2$. We can see this by drawing a circle. About any point y in the interior of the unit disk we may draw a circle which does not meet the boundary of $B(0, 1)$. The radius of this circle, which we can take to be $1 - d(x, y)$ if we like, serves as ϵ in the definition of an open set.

Example The previous example is not special. It is easy to use the same reasoning to show that $B(x, \epsilon)$ is open for all $x \in \mathbb{R}^n$, $\epsilon > 0$.

Example Let $\{U_i\}_{i \in I}$ be a collection of open sets in $\mathbb{R}^n$. Then the union $U = \cup_{i \in I} U_i$ is an open set in $\mathbb{R}^n$ as well. The reasoning for this: let $x \in U$. By the definition of union we can find U_i containing x. Then find $\epsilon > 0$ with $B(x, \epsilon) \subset U_i$, by the fact that U_i is open. As the union of sets contains each of those sets, we have $B(x, \epsilon) \subset U_i \subset U$. Thus U is open.

Example Suppose that U and V are open sets in $\mathbb{R}^n$. Then $U \cap V$, the intersection of the two sets U and V is open as well. The reasoning for this: if $x \in U \cap V$, then we can find ϵ_1 so that $B(x, \epsilon_1) \subset U$ and ϵ_2 so that $B(x, \epsilon_2) \subset V$. Whichever of the two radii is the smaller, say ϵ_1, we take that for ϵ in the definition and see that $B(x, \epsilon) \subset U \cap V$. It is not true that if you take the arbitrary intersection of open sets you produce an open set. For example, if you take the intersection of the open sets $(-1/n, 1/n)$ in $\mathbb{R}$ you get the one point set $\{0\}$, which is not open.

Theorem 1. *Let $U \subset \mathbb{R}^n$ be non-empty. Then U is an open subset of $\mathbb{R}^n$ if and only if there is a collection of open balls $B(x_i, \epsilon_i)$, where $i \in I$ belongs to some index set, such that $X = \cup_{i \in I} B(x_i, \epsilon_i)$.*

Proof. Once you know that open balls are open subsets of $\mathbb{R}^n$, the previous statement about unions gives that a union of open balls is an open subset of $\mathbb{R}^n$. If we have an open subset

$U \subset \mathbb{R}^n$, then for each $x \in U$ pick $B(x, \epsilon_x) \subset U$. Then the collection $\{B(x, \epsilon_x)\}_{x \in U}$ is a collection of open balls satisfying

$$U = \cup_{x \in U} B(x, \epsilon_x),$$

because $x \in B(x, \epsilon_x)$ and $B(x, \epsilon_x) \subset U$. □

Now we are in a position to give a new "global" definition of continuity:

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function. Then f is continuous if and only if for every open set $U \subset \mathbb{R}^m$, the pre-image $f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}$ is an open set in $\mathbb{R}^n$.*

Notice that this theorem does not make specific reference to a given point, only to the collection of open sets that we described with open balls and unions.

Proof. Suppose that f is continuous. Let $U \subset \mathbb{R}^m$ be an open set. If $f^{-1}(U)$ is $\emptyset$, then it is open by definition (the empty set, recall, is always open). Suppose instead that $x \in f^{-1}(U)$. Then $f(x) \in U$, and as U is an open subset of $\mathbb{R}^m$ we can select $\epsilon > 0$ so that $B(f(x), \epsilon) \subset U$. Now f is a continuous function, so given this ϵ, we can find $\delta > 0$ so that $||x - y|| < \delta$ implies that $||f(x) - f(y)|| < \epsilon$. (This is just using the definition of continuity). This means that if $y \in B(x, \delta)$, then $f(y) \in B(f(x), \epsilon) \subset U$. So if $y \in B(x, \delta)$, $f(y) \in U$. That means that $B(x, \delta) \subset f^{-1}(U)$. Thus about the point $x \in f^{-1}(U)$ we have found an open ball $B(x, \delta)$ contained entirely in $f^{-1}(U)$, which is precisely what we needed in order for $f^{-1}(U)$ to be open. Thus if f is continuous, $f^{-1}(U)$ is open for all open sets $U \subset \mathbb{R}^m$.

Now suppose that $f^{-1}(U)$ is open for all open sets $U \subset \mathbb{R}^m$. We want to show that f is continuous at every $x \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ and let $\epsilon > 0$. We seek $\delta > 0$ so that $||x - y|| < \delta$ implies that $||f(x) - f(y)|| < \epsilon$.

The set $B(f(x), \epsilon) \subset \mathbb{R}^m$ is open, because open balls are always open. Thus the pre-image $f^{-1}(B(f(x), \epsilon))$ is open as well, and as $f(x) \in B(f(x), \epsilon)$, we have $x \in f^{-1}(B(f(x), \epsilon))$. Thus by the openness of this latter set, we can pick $\delta > 0$ so that $B(x, \delta) \subset f^{-1}(B(f(x), \epsilon))$.

Thus $||x - y|| < \delta$ implies that $y \in f^{-1}(B(f(x), \epsilon))$. Then $f(y) \in B(f(x), \epsilon)$. Thus $||f(x) - f(y)|| < \epsilon$. So f is continuous at x. As x was an arbitrary point of $\mathbb{R}^n$, we have shown that f is continuous. Thus if $f^{-1}(U)$ is open for every open subset U of $\mathbb{R}^m$, the function f is continuous. □

This allows us to present a more concise definition of continuity for functions $f : \mathbb{R}^n \to \mathbb{R}^m$, albeit one that requires a bit more terminology than the original: **Definition** (Revised definition of continuity) A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuous* if $f^{-1}(U)$ is open for every open subset U of $\mathbb{R}^m$.

Whew. All of this is motivation and groundwork for the notion of a topological space, which is a massive generalization of the ideas we've presented here. The upshot is: a topological space encodes the same ideas of "nearness" as the distance function $d(x, y) = ||x - y||$ on $\mathbb{R}^n$, but without the numbers. A topological space does not have to consist of n-tuples of real numbers; instead, it is simply a set X. Along with X collection of *open subsets* of X which behave much like the open subsets in $\mathbb{R}^n$. Instead of the open balls out of which we built every non-empty open set in $\mathbb{R}^n$, in a topological space we use a distinguished collection of open sets called a *basis*. A function $f : X \to Y$ between two topological spaces is continuous if it satisfies the last definition of continuity, i.e. if $f^{-1}(U)$ is open for every open set U in Y.

As you can see from the preceding, the notion of sets, subsets, functions, pre-images, and so forth are all vital. Thus, the first homework assignment will deal with manipulations of sets.