MATH 54 - LECTURE 10

DAN CRYTSER

The Universal Mapping Property

First we note that each of the projection mappings $\pi_i : \prod_j X_j \to X_i$ is continuous when $\prod_i X_i$ is given the product topology (also if the product is given the box topology, but this is not relevant).

Lemma 1. *Let $X = \prod_j X_j$, and let $\pi_i : X \to X_i$ be the canonical projection onto the ith factor, i.e. $\pi_i(f) = f(i)$. If X is equipped with the product topology, then π_i is continuous.*

Proof. Suppose that $U_i \subset X_i$ is continuous. We show that $\pi_i^{-1}(U_i)$ is open in the product topology. But $\pi_i^{-1}(U_i)$ is simply the basis element in the product topology $\prod V_j$ which has $V_i = U_i$ and all other $V_j = X_j$. This is open as it is a basis element in the product topology. □

The "universal mapping property" is a tool for converting a collection of continuous maps into one "large" continuous map which holds all of their information. We call it a "universal mapping property" because the recipe which produces this large map does not vary from collection to collection; we are always following the same algorithm to produce the large map, and it is always unique. This is made precise in the following theorem, which we state a bit differently from Munkres (owing to that worldliness and sophistication which we, as a class, tend to possess):

Theorem 1. *Let A be a space, let $\{X_i\}_{i \in I}$ be a family of spaces, and suppose that we are given, for each $i \in I$, a continuous mapping $f_i : A \to X_i$. Equip $\prod_i X_i$ with the product topology. Then there is a unique continuous map $f : A \to \prod_i X_i$ such that $\pi_i \circ f = f_i$ for all $i \in I$. Conversely, if $f : A \to \prod_i X_i$ is continuous, then each of the maps $f_i = \pi_i \circ f$ is continuous.*

Proof. The requirement that $\pi_i \circ f = f_i$ forces $f(a)(i) = f_i(a)$ (recall that $f(a) \in \prod_i X_i$ and so $f(a)$ is a function defined on the i). Thus if the function exists, it is unique. We show that f which carries to a to $f(a)$ is continuous. Let $U = \prod_i U_i$ be a basis element of $\prod_i X_i$ in the product topology, so that $U_i = X_i$ for all but finitely many i. Then $f^{-1}(U) = \cap_i f_i^{-1}(U_i)$, as you can check. This is an open set. You might say "Dan, stop this. You are attempting to claim that the arbitrary intersection of open sets is open, and even a fourth-grader knows this to be false." But I know that all but finitely many of the U_i are equal to their total spaces X_i, so that $f_i^{-1}(U_i) = A$ for all but finitely many i, and so the intersection is really taken over a finite collection of open subsets of A (intersecting finitely many subsets of A with a bunch of copies of A does not change anything – the intersection is the same). Thus $f^{-1}(U)$ is open in A, and so f is continuous.

If the function f is taken to exist, then the functions $f_i = \pi_i \circ f$ are all continuous because they are compositions of continuous functions. □

Metric Spaces

We introduced the course with a review of Euclidean space, and after we defined topological spaces we were able to see that

Euclidean spaces $\subset$ **Topological Spaces**.

That is, every Euclidean space is naturally a topological space, with topology generated by the basis of open balls. In this section we insert a third category of space between these two, a category which contains many of the most interesting types of topological spaces.

We get

Euclidean spaces $\subset$ **Metric spaces** $\subset$ **Topological Spaces**.

Definition Let X be a set. A function $d : X \times X \to [0, \infty)$ is called a *metric* on X if the following hold:

(1) (Positive definiteness): We have $d(x, y) = 0$ if and only if $x = y$.
(2) (Symmetry): For all $x, y \in X$, we have $d(x, y) = d(y, x)$.
(3) (Triangle Inequality): For all $x, y, z \in X$, we have $d(x, y) + d(y, z) \geq d(x, z)$.

The pair (X, d) is called a metric space.

Example The premier examples of a metric spaces are the Euclidean space $\mathbb{R}^n$ for $n > 0$. The metric is given by $d(x, y) = ||x - y||$ for $x, y \in \mathbb{R}^n$, where $||x||$ denotes the norm of the vector x. Proving the first two properties of the metric is not complicated, but showing that the triangle inequality holds requires something called the Cauchy-Schwarz inequality, something that you will look at in the HW.

Example Let X be a set, let $d : X \times X \to \{0, 1\}$ be given by

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

Then you can check this is a metric on X. We call it the discrete metric on X.

Just as we generated the Euclidean topology on $\mathbb{R}^n$ with the basis of open balls, we can generate a topology on a metric space X using the open balls defined by the metric.

Definition Let (X, d) be a metric space. Let $x \in X$ and $\epsilon > 0$. Then the open ball centered at x with radius ϵ is the set $B_d(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$. When there is no ambiguity as to which metric is in use, we suppress the d in the notation for the open ball and simply write $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$.

Now that we have the open balls, it's just a hop skip and a jump to define the topology.

Definition Let (X, d) be a metric space. We define the *metric topology* on X to be the topology generated by the basis $\mathcal{B} = \{B(x, \epsilon) : x \in X, \epsilon > 0\}$. Thus a set $U \subset X$ is open in the metric topology if for any $x \in U$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. This is also called the *topology induced by d*.

NOTE: we have implicitly claimed that the collection $\mathcal{B} = \{B(x, \epsilon) : x \in X, \epsilon > 0\}$ actually is a basis. Recall that in order to prove this, we have to check the covering and intersection properties. As $x \in B(x, \epsilon)$ for any $\epsilon > 0$ by the positive definite axiom, the covering property is fulfilled. Suppose that $z \in B(x, \epsilon) \cap B(y, \epsilon')$. Let $\epsilon'' = \min\{\epsilon - d(x, z), \epsilon' - d(y, z)\}$. You can check that $B(z, \epsilon'')$ lies in the intersection $B(x, \epsilon) \cap B(y, \epsilon')$.

Example The topology on $\mathbb{R}^n$ generated by the metric $d(x, y) = ||x - y||$ is the Euclidean topology.

Example The topology generated by the discrete metric is (surprise, surprise) the discrete topology on X, as for each $x \in X$ the open ball $B(x, 1) = \{x\}$ is open in the topology generated by the discrete metric.

Topological Equivalence of Metrics

Definition Two metrics d and d' on a set X are said to be topologically equivalent if they generate the same topology.

Definition Let X be a set, let d and d' be two metrics on X. Then d and d' are topologically equivalent if their metric topologies are the same.

Theorem 2. *Suppose that X is a set, d and d' are two metrics on X. Let τ be the metric topology on X generated by d and let τ' be the topology generated by d'. Then TFAE:*

(1) $\tau \subset \tau'$, that is the topology generated by d' is finer than the topology generated by d.
(2) For any $x \in X$ and any $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Proof. Suppose that 1 holds. Consider the set $B_d(x, \epsilon)$. This is an open set in τ, so it must also be an open set in τ'. As $x \in B_d(x, \epsilon)$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$ (by definition of the metric topology generated by d').

Suppose that 2 holds. Let $U \in \tau$. If $U = \emptyset$ then $U \in \tau'$. Suppose that $U \neq \emptyset$. Let $x \in U$. As $U \in \tau$ and $x \in U$, we can find $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$. As 2 holds, we find $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon) \subset U$. Therefore about any point $x \in U$, there is a ball $B_{d'}(x, \delta) \subset U$. So U is open in the metric topology generated by d'.

□

We give an example of this:

Definition The square metric on $\mathbb{R}^n$ is defined for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ to be

$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

The open balls in the square metric look like open intervals/squares/cubes in $\mathbb{R}/\mathbb{R}^2/\mathbb{R}^3$.

Theorem 3. *The metrics ρ and d on $\mathbb{R}^n$ are topologically equivalent; that is, every set which is open in the topology induced by ρ is open in the topology induced by d, and conversely.*

Proof. First we prove a few inequalities between the metrics. Let $x, y \in \mathbb{R}^n$, and suppose that $|x_i - y_i| = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$. Then $|x_k - y_k| \leq |x_i - y_i|$ for each $k = 1, \ldots, n$. Squaring each side of this equation preserves the order, $(x_k - y_k)^2 \leq (x_i - y_i)^2$. Note that

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \leq \sqrt{(x_i - y_i)^2 + \ldots + (x_i - y_i)^2} = \sqrt{n}\rho(x, y).$$

Also $(x_i - y_i)^2 \leq (x_1 - y_1)^2 + \ldots + (x_n - y_n)^2$, so on taking square roots we obtain

$$\rho(x, y) = |x_i - y_i| \leq \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} = d(x, y).$$

Thus $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$.

Now it is easy to check that $B_d(x, \epsilon) \subset B_\rho(x, \epsilon)$ and that $B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \subset B_d(x, \epsilon)$ for all $x \in \mathbb{R}^n$ and $\epsilon > 0$. Thus the topologies they generate are finer than one another, hence equal.

□

Properties of Metric Spaces

We obtain strong theorems for metric spaces in comparison to the rest of topological spaces.

The first of these results enables us to determine the closure of a set using the metric, or using sequences. Note that sequences do not suffice to determine closures in general topological spaces (see Munkres, Example 1, page 132).

Theorem 4. *Let X be a metric space endowed with the metric topology (henceforth this is implicit in all discussion of metric spaces). Let $x \in X$ and $A \subset X$. Then TFAE:*

(1) $x \in \overline{A}$.
(2) For all $\epsilon > 0$, there exists $a \in A$ such that $d(x, a) < \epsilon$.
(3) There is a sequence (a_n) of points of A that converges to x.

Proof. ($1 \to 2$): Suppose that $x \in \overline{A}$, and let $\epsilon > 0$. Then $B(x, \epsilon)$ is a nbhd of x, and so there exists $a \in A \cap B(x, \epsilon)$. But this is just the assertion (2).

($2 \to 3$): Suppose that the condition (2) holds. We construct a sequence of points of A which converges to x. For $n \in \mathbb{N}$, pick $a_n \in A$ such that $d(x, a_n) < 1/n$. Claim: the sequence (a_n) converges to x. For if U is a nbhd of x, we can pick $\epsilon > 0$ such that $B(x, \epsilon) \subset U$, as U is open. Then we pick N such that $1/N < \epsilon$. Then for all $n > N$, we have $1/n < 1/N < \epsilon$, so that $a_n \in U$ for all $n > N$. Thus $a_n \to x$.

($3 \to 1$): Suppose that the sequence (a_n) converges to x. Let U be a nbhd of x. We can pick $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n \in U$. Then $a_{N+1} \in U \cap A$. Thus every nbhd U of x intersects A, so that $x \in \overline{A}$. □

Note that the implication ($3 \to 1$) above holds for any topological space, but not conversely. If we equip $\prod_{n=1}^{\infty} \mathbb{R}$ with the box topology, then the point $0 = (0, 0, 0, 0, \ldots)$ is in the closure of $\prod_{n=1}^{\infty}(0, \infty)$, yet no sequence of $\prod_{n=1}^{\infty} \mathbb{R}$ converges to 0 in the box topology. The other implications do not make sense in a general topological (non-metric) space.

Theorem 5. *If X is a metric space, then X is Hausdorff.*

Proof. Let $x \neq y$ in X. Then $d = d(x, y) > 0$. Let $U = B(x, d/2)$ and $V = B(y, d/2)$. Note that if $z \in X$, then $d = d(x, y) \leq d(x, z) + d(y, z) = d(x, z) + d(y, z)$. Thus $d(x, z) \geq d - d(y, z)$. Thus if $z \in B(y, d/2)$, then $d(y, z) < d/2$, and $d - d(y, z) > d/2$. Thus if $z \in B(y, d/2)$, $d(x, z) \geq d/2$, so that $B(y, d/2) \cap B(x, d/2) = \emptyset$. As these balls are basis elements for the metric topology, they are in particular open sets in the metric topology. Thus the metric topology is a Hausdorff space.

□

Theorem 6. *Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. Then f is continuous if and only if for each $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \epsilon$.*

Proof. This is very similar to the proof in the Euclidean case.

Suppose that f is continuous. Let $x \in X$ and $\epsilon > 0$. Then $B_{d_Y}(f(x), \epsilon)$ is an open set in Y. Thus the preimage $f^{-1}(B_{d_Y}(f(x), \epsilon))$ is an open set in X. As $f(x) \in B_{d_Y}(f(x), \epsilon)$, we have $x \in f^{-1}(B_{d_Y}(f(x), \epsilon))$. As the preimage is open in X, we can find $\delta > 0$ such that $B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \epsilon))$. Then $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \epsilon$.

Suppose that the ϵ-δ condition holds. Let $V \subset Y$ be open. We show that the preimage $f^{-1}(V)$ is open. If $f^{-1}(V)$ is empty then we are done. Suppose that $x \in f^{-1}(V)$. Then $f(x)$ is an element of the open set V, so that we can produce $\epsilon > 0$ such that $B_{d_Y}(f(x), \epsilon) \subset V$. Produce $\delta > 0$ such that $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \epsilon$. Then $B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \epsilon))$. Thus about any point of the preimage we have found an open ball contained in the preimage, so that the preimage is open.

□