MATH 54 - LECTURE 11

DAN CRYTSER

Introduction

We revisit the notion of a metric space and explore some of their properties.

Topological Equivalence of Metrics

Definition Two metrics d and d' on a set X are said to be topologically equivalent if they induce the same topology.

Theorem 1. *Suppose that X is a set, d and d' are two metrics on X. Let τ be the metric topology on X generated by d and let τ' be the topology generated by d'. Then TFAE:*

(1) $\tau \subset \tau'$, that is the topology generated by d' is finer than the topology generated by d.
(2) For any $x \in X$ and any $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$.

Proof. Suppose that 1 holds. Consider the set $B_d(x, \epsilon)$. This is an open set in τ, so it must also be an open set in τ'. As $x \in B_d(x, \epsilon)$, there exists a $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$ (by definition of the metric topology generated by d').

Suppose that 2 holds. Let $U \in \tau$. If $U = \emptyset$ then $U \in \tau'$. Suppose that $U \neq \emptyset$. Let $x \in U$. As $U \in \tau$ and $x \in U$, we can find $\epsilon > 0$ such that $B_d(x, \epsilon) \subset U$. As 2 holds, we find $\delta > 0$ such that $B_{d'}(x, \delta) \subset B_d(x, \epsilon) \subset U$. Therefore about any point $x \in U$, there is a ball $B_{d'}(x, \delta) \subset U$. So U is open in the metric topology generated by d'.

□

We give an example of this:

Definition The square metric on $\mathbb{R}^n$ is defined for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ to be

$$\rho(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}.$$

The open balls in the square metric look like open intervals/squares/cubes in $\mathbb{R}/\mathbb{R}^2/\mathbb{R}^3$.

Theorem 2. *The metrics ρ and d on $\mathbb{R}^n$ are topologically equivalent; that is, every set which is open in the topology induced by ρ is open in the topology induced by d, and conversely.*

Proof. First we prove a few inequalities between the metrics. Let $x, y \in \mathbb{R}^n$, and suppose that $|x_i - y_i| = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$. Then $|x_k - y_k| \leq |x_i - y_i|$ for each $k = 1, \ldots, n$. Squaring each side of this equation preserves the order, $(x_k - y_k)^2 \leq (x_i - y_i)^2$. Note that

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \leq \sqrt{(x_i - y_i)^2 + \ldots + (x_i - y_i)^2} = \sqrt{n}\rho(x, y).$$

Also $(x_i - y_i)^2 \leq (x_1 - y_1)^2 + \ldots + (x_n - y_n)^2$, so on taking square roots we obtain

$$\rho(x, y) = |x_i - y_i| \leq \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} = d(x, y).$$

Thus $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$.

Now it is easy to check that $B_d(x, \epsilon) \subset B_\rho(x, \epsilon)$ and that $B_\rho(x, \frac{\epsilon}{\sqrt{n}}) \subset B_d(x, \epsilon)$ for all $x \in \mathbb{R}^n$ and $\epsilon > 0$. Thus the topologies they generate are finer than one another, hence equal.

□

Metric Spaces Are Hausdorff Spaces

Theorem 3. *If X is a metric space, then X is Hausdorff.*

Proof. Let $x \neq y$ in X. Then $d = d(x, y) > 0$. Let $U = B(x, d/2)$ and $V = B(y, d/2)$. Note that if $z \in X$, then $d = d(x, y) \leq d(x, z) + d(y, z) = d(x, z) + d(y, z)$. Thus $d(x, z) \geq d - d(y, z)$. Thus if $z \in B(y, d/2)$, then $d(y, z) < d/2$, and $d - d(y, z) > d/2$. Thus if $z \in B(y, d/2)$, $d(x, z) \geq d/2$, so that $B(y, d/2) \cap B(x, d/2) = \emptyset$. As these balls are basis elements for the metric topology, they are in particular open sets in the metric topology. Thus the metric topology is a Hausdorff space.

□

Continuous Functions between Metric Spaces

Theorem 4. *Let $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces. Then f is continuous if and only if for each $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d_X(x, x') < \delta$ implies $d_Y(f(x), f(x')) < \epsilon$.*

Proof. This is very similar to the proof in the Euclidean case.

Suppose that f is continuous. Let $x \in X$ and $\epsilon > 0$. Then $B_{d_Y}(f(x), \epsilon)$ is an open set in Y. Thus the preimage $f^{-1}(B_{d_Y}(f(x), \epsilon))$ is an open set in X. As $f(x) \in B_{d_Y}(f(x), \epsilon)$, we have $x \in f^{-1}(B_{d_Y}(f(x), \epsilon))$. As the preimage is open in X, we can find $\delta > 0$ such that $B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \epsilon))$. Then $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \epsilon$.

Suppose that the ϵ-δ condition holds. Let $V \subset Y$ be open. We show that the preimage $f^{-1}(V)$ is open. If $f^{-1}(V)$ is empty then we are done. Suppose that $x \in f^{-1}(V)$. Then $f(x)$ is an element of the open set V, so that we can produce $\epsilon > 0$ such that $B_{d_Y}(f(x), \epsilon) \subset V$. Produce $\delta > 0$ such that $d_X(x, x') < \delta$ implies that $d_Y(f(x), f(x')) < \epsilon$. Then $B_{d_X}(x, \delta) \subset f^{-1}(B_{d_Y}(f(x), \epsilon))$. Thus about any point of the preimage we have found an open ball contained in the preimage, so that the preimage is open.

□

Theorem 5. *Let X be a metric space endowed with the metric topology (henceforth this is implicit in all discussion of metric spaces). Let $x \in X$ and $A \subset X$. Then TFAE:*

(1) $x \in \overline{A}$.
(2) For all $\epsilon > 0$, there exists $a \in A$ such that $d(x, a) < \epsilon$.
(3) There is a sequence (a_n) of points of A that converges to x.

Proof. $(1 \to 2)$: Suppose that $x \in \overline{A}$, and let $\epsilon > 0$. Then $B(x, \epsilon)$ is a nbhd of x, and so there exists $a \in A \cap B(x, \epsilon)$. But this is just the assertion (2).

$(2 \to 3)$: Suppose that the condition (2) holds. We construct a sequence of points of A which converges to x. For $n \in \mathbb{N}$, pick $a_n \in A$ such that $d(x, a_n) < 1/n$. Claim: the sequence (a_n) converges to x. For if U is a nbhd of x, we can pick $\epsilon > 0$ such that $B(x, \epsilon) \subset U$, as U is open. Then we pick N such that $1/N < \epsilon$. Then for all $n > N$, we have $1/n < 1/N < \epsilon$, so that $a_n \in U$ for all $n > N$. Thus $a_n \to x$.

$(3 \to 1)$: Suppose that the sequence (a_n) converges to x. Let U be a nbhd of x. We can pick $N \in \mathbb{N}$ such that for all $n > N$ we have $a_n \in U$. Then $a_{N+1} \in U \cap A$. Thus every nbhd U of x intersects A, so that $x \in \overline{A}$. □

Note that the implication $(3 \to 1)$ above holds for any topological space, but not conversely. If we equip $\prod_{n=1}^{\infty} \mathbb{R}$ with the box topology, then the point $0 = (0, 0, 0, 0, \ldots)$ is in the closure of $\prod_{n=1}^{\infty}(0, \infty)$, yet no sequence of $\prod_{n=1}^{\infty} \mathbb{R}$ converges to 0 in the box topology. The other implications do not make sense in a general topological (non-metric) space.

This theorem enables to state a result characterizing continuity of functions between metric spaces in terms of sequences, which are often convenient to work with.

Theorem 6. *Let $f : X \to Y$ be a function, where X is a metric space and Y is a topological space. Then f is continuous if and only if given a convergent sequence $x_n \to x$ in X, the image sequence $f(x_n)$ converges to $f(x)$ in Y.*

Proof. Suppose that f is continuous, and let $x_n \to x$ in X. Let V be a nbhd of $f(x)$ in Y. We show that $(f(x_n))$ eventually lies within V. As f is continuous, the preimage $f^{-1}(V)$ is a nbhd of x. Since $x_n \to x$, we can select N such that $n > N$ implies that $x_n \in f^{-1}(V)$. But then $n > N$ implies that $f(x_n) \in U$. Thus $f(x_n) \to f(x)$.

Now suppose that the sequence condition holds. We show that f is continuous by proving that $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$. Let $A \subset X$. Suppose that $y = f(x)$ for some $x \in \overline{A}$. Then by the previous lemma there exists a sequence (a_n) of points of A such that $a_n \to x$. Then $f(a_n) \to f(x)$, so that a sequence of points of $f(A)$ converges to $f(x)$. Thus the sequence $f(a_n)$ converges to $f(x)$, which implies as noted that $f(x) \in \overline{f(A)}$. □

Continuity of Algebraic Operations, Algebraic Operations on Continuous Functions

Lemma 1. *The addition, subtraction, and multiplication operations are continuous from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient op is cts from $\mathbb{R} \times (\mathbb{R} - \{0\})$ to $\mathbb{R}$.*

Proof. You are responsible for proving this in the HW for the week. □

Theorem 7. *Suppose that $f, g : X \to \mathbb{R}$ are continuous functions from some topological space X into the real line. Then the sum $f + g$ and the product fg are both continuous functions from X to $\mathbb{R}$. If g is never 0, then the quotient f/g is continuous.*

Proof. The map $x \to (f(x), g(x))$ is continuous into $\mathbb{R} \times \mathbb{R}$ as each of the coordinate functions is continuous. As the sum and product maps are continuous from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, the map $x \to (f(x), g(x)) \to f(x) + g(x)$ and $x \to (f(x), g(x)) \to f(x)g(x)$ are compositions of continuous functions, hence continuous.

The case when g excludes 0 from its range is similar.

□