

# Math 54 - Lecture 12: Quotient Spaces

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## Introduction

First, we tie up a loose end, showing that a map  $f : X \rightarrow Y$  from a metric space to a topological space is continuous if and only if  $f(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$ .

The rest of the lecture is a brief overview of the notion of a quotient topological space. A quotient space is a recipe: equivalence relation  $\sim$  on a set  $X$  + topology on  $X$  = topology on  $X/\sim$  the collection of equivalence classes. Quotient spaces are used frequently in other areas of mathematics to make new topological spaces.

## Continuity

**Theorem 1.** *Suppose that  $f : X \rightarrow Y$  is a function from  $X$  a metric space to  $Y$  a topological space. Then  $f$  is continuous iff  $f(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$ .*

*Proof.* We already proved the necessity of the sequence condition. We prove that it implies continuity. We show that for any subset  $A \subset X$ , the inclusion  $f(\overline{A}) \subset \overline{f(A)}$  holds. Suppose that  $x \in \overline{A}$  and let  $y = f(x)$ . We show that  $y \in \overline{f(A)}$ . Then by a result from Monday, we know that there is a sequence  $(a_n) \subset A$  such that  $a_n \rightarrow x$ . Then  $f(a_n) \rightarrow f(x)$ . Thus the sequence  $f(a_n) \subset f(A)$  converges to  $f(x)$ , and  $f(x) \in \overline{f(A)}$ . Thus  $f(\overline{A}) \subset \overline{f(A)}$ , and  $f$  is continuous.  $\square$

## Equivalence Relations and Equivalence Classes

**Definition** If  $X$  is a set, then an equivalence relation  $\sim \subset X \times X$  is a relation such that the following hold:

1.  $x \sim x$  for all  $x \in X$
2. if  $x \sim y$  then  $y \sim x$ .
3. if  $x \sim y$  and  $y \sim z$  then  $x \sim z$

**Example** On the integers, the relation  $x \sim y$  if and only if  $x - y$  is even is an equivalence relation. The relation  $x \sim y$  if and only if  $x - y$  is odd is NOT an equivalence relation, as it fails the third axiom ( $1 \sim 2$  and  $2 \sim 3$ , but  $1 \not\sim 3$ ).

We use equivalence relations to split spaces up into sub-classes, called “equivalence classes.”

**Definition** Let  $\sim$  be an equivalence relation on a set  $X$ . If  $x \in X$ , then  $[x]_{\sim}$ , the equivalence class containing  $x$ , is defined as

$$[x]_{\sim} = \{y \in X : y \sim x\}.$$

The collection  $\{[x] : x \in X\}$  is denoted by  $X/\sim$ .

**Example** The equivalence relation of the previous example yields  $\mathbb{Z}/\sim = \{[0], [1]\}$ , because each number is either even, and hence in  $[0]$ , or odd, and hence in  $[1]$ .

**Definition** If  $X$  is a set and  $\sim$  is an equivalence relation on  $X$ , then the map  $\pi : X \rightarrow X/\sim$  defined by  $\pi(x) = [x]$  is called the projection map.

## Placing a Topology on $X/\sim$

We often have a topological space  $X$  and some easily defined equivalence relation  $\sim$  on  $X$ . This gives us a set  $X/\sim$ , but doesn't yet tell us how to define a topology on  $X/\sim$ . The main requirement that we place on the topology of  $X/\sim$  is that the map  $\pi : X \rightarrow X/\sim$  is continuous.

**Example** If we give  $X/\sim$  the indiscrete/concrete topology, which only has the null and total sets as open sets, then the map  $\pi : X \rightarrow X/\sim$  is continuous. This is useless, though, because no information is contained in the concrete topology.

**Definition** Let  $X$  be a set with an equivalence relation  $\sim$ . Then the *quotient topology* on  $X/\sim$  is defined by the rule: subset  $U \subset X/\sim$  is open if and only if the preimage  $\pi^{-1}(U)$  is open.

## A Universal Mapping Property for the Quotient Space

The quotient space is similar to the product space in that it allows us to manufacture continuous maps out of other continuous maps. The central question is this: given a continuous map  $f : X \rightarrow Y$ , when can we define a map  $\tilde{f} : X/\sim \rightarrow Y$  such that  $\tilde{f} \circ \pi = f$ . That is, when can we define  $\tilde{f}$  on the quotient space and still get  $\tilde{f}([x]) = f(x)$ ?

We can get a necessary condition right away: if  $[x] = [x']$ , that is, if  $x \sim x'$ , then we must have  $f(x) = f(x')$ . For if such a  $\tilde{f}$  exists, then we have  $f(x) = \tilde{f}([x]) = \tilde{f}([x']) = f(x')$ . We say that  $f$  is *constant on classes* if it satisfies this rule.

We will see that this necessary condition is also, wonderfully, a sufficient condition: if  $f(x) = f(x')$  whenever  $x \sim x'$ , then we can produce  $\tilde{f}$  satisfying the requirement above.

**Theorem 2.** *Let  $X$  be a topological space with  $X/\sim$  a quotient space. If  $f : X \rightarrow Y$  is a continuous function from  $X$  into some topological space  $Y$ , and if  $f(x) = f(x')$  whenever  $x \sim x'$ , then there exists a (unique) continuous function  $\tilde{f} : X/\sim \rightarrow Y$  such that  $\tilde{f} \circ \pi = f$ . Specifically,  $\tilde{f}([x]) = f(x)$ .*

*Proof.* That  $\tilde{f}([x]) = f(x)$  defines a function on  $X/\sim$  follows from the fact that  $f$  is constant on the classes. The uniqueness follows from  $\tilde{f} \circ \pi = f$ . We show that  $\tilde{f}$  is continuous. Let  $U \subset Y$  be open. Then  $\tilde{f}^{-1}(U) = \{[x] : \tilde{f}([x]) \in U\} = \{[x] : f(x) \in U\}$ . We check that this set, call it  $V$ , is open in  $X/\sim$ . But  $\pi^{-1}(V) = \{x : f(x) \in U\} = f^{-1}(U)$ . As  $f$  is continuous,  $V$  is an open subset of  $X$ . Thus  $\pi^{-1}(V)$  is open in  $X$ , which is the precise condition we need to show that  $V = \tilde{f}^{-1}(U)$  is open in  $X/\sim$ .  $\square$

## An Example

We use the universal mapping property of the quotient space to show that if you identify the two endpoints of the unit interval  $[0, 1]$ , the resulting quotient space is homeomorphic to the subspace  $\mathbb{T} \subset \mathbb{R}^2$ .

**Example** Let  $X = [0, 1]$  (with std. top.) and define  $x \sim y$  if  $x = y$  or if  $x = 0$  and  $y = 1$  or if  $x = 1$  and  $y = 0$ . (So, the only distinct points of  $X$  which are equivalent are 0 and 1). Let  $\mathbb{T}$  denote the unit circle, as a subspace of  $\mathbb{R}^2$ . I claim that we can define a homeomorphism from  $X/\sim$  to  $\mathbb{T}$ . First we define  $f : [0, 1] \rightarrow \mathbb{T}$  by  $f(x) = (\cos(2\pi x), \sin(2\pi x))$ . We get a map  $F : X/\sim \rightarrow \mathbb{T}$  because  $f$  is constant on the fibers (the only thing to check is  $(\cos(0), \sin(0)) = (\sin(2\pi), \cos(2\pi))$ ). This map  $F$  is our required homeomorphism. For  $F$  is continuous by the universal mapping property. You can check that  $F$  is a bijection, because the curve  $f : [0, 1] \rightarrow \mathbb{T}$  is onto and the map  $F$  is injective. We only need to check that the map  $F^{-1} : \mathbb{T} \rightarrow X/\sim$  is continuous. Checking that the inverse function is continuous is the same as checking that the map  $F : X/\sim \rightarrow \mathbb{T}$  is open, that is, the image  $F(U)$  of any open subset  $U \subset X/\sim$  is open. We

- (a)  $U \subset X/\sim$  does not contain  $[0]$ . Then  $U$  is a union of open intervals in  $[0, 1]$ . The image  $F(U)$  is a bunch of "open intervals" in  $\mathbb{T}$ . Each of these is open, so  $F(U)$  is open.
- (b)  $[0] \in U$ . As  $U$  is open in  $X/\sim$ , we must have  $\pi^{-1}(U)$  open and containing  $[0] = \{0, 1\}$ . Thus it contains a small half-open interval around 0 and another around 1. The image contains the point  $(1, 0)$  and a small OPEN interval (taking the union of the images of these two half-open intervals). The rest of the points have nbhds contained in the image  $F(U)$  as well, so that  $F(U)$  is open.