Math 54 - Lecture 13: Connected Spaces

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Introduction

We introduce connected spaces. Then we show that intervals in the real line are all connected. We show that the image of a connected space is connected, then use this to prove the intermediate value theorem.

Connectedness

The idea of connectedness is simple: some spaces can be partitioned into disjoint open sets, and others cannot. Those which cannot be so partitioned are *connected*.

Example The space $[0,1] \cup [2,3]$ is made out of two pieces. The subset $[0,1]$ and $[2,3]$ are separate, in the sense that, on zooming in on any point of $[0,1]$, I don't see any points of $[2,3]$, and vice versa. More precisely, $[0,1]$ does not contain any limit points of $[2,3]$, and vice versa.

Example The space $[0,2]$ consists of one piece. You can try to write it as $[0,1] \cup (1,2]$, but these spaces don't enjoy the same property. Specifically, 1 is a limit point of $(1,2]$.

Definition Let X be a space. A separation of X is a pair U, V of disjoint nonempty open subsets of X such that $X = U \cup V$. If no separation of X exists, then we say that X is connected.

Example Any set with the indiscrete topology is trivially connected, because there are no proper nonempty open subsets.

Example The real line with the cofinite topology is connected, because there are no pairs of disjoint non-empty open subsets.

The Real Line

The most important example of a connected space is the real line with the standard topology, and every interval contained therein in the subspace topology.

Theorem 3. *Let $I \subset \mathbb{R}$ be an interval of real numbers. Then I is connected in the subspace topology.*

Proof. We contradict. Let $I = A \cup B$, where A and B are disjoint nonempty open subsets, and suppose that $a \in A$ and $b \in B$ such that $a < b$. Notice that as I is an interval, we have $[a, b] \subset I$. Let $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$. Note that these are both open subsets of $[a, b]$ in the subspace topology, and note that $A_0 \cup B_0 = [a, b]$. As $\mathbb{R}$ has the least upper bound property, and A_0 is bounded above by b, there must exist a least upper bound $c \in \mathbb{R}$ for A_0. As $a \leq c$, and $c \leq b$ (as b is also an upper bound for A_0), we have $c \in [a, b]$. We prove that c is in neither A_0 nor B_0, contradicting $[a, b] = A_0 \cup B_0$.

1. Suppose that $c \in B_0$. As $c \neq a$ and B_0 is open, there must be some $\epsilon > 0$ such that $(c - \epsilon, c] \subset B_0$. But then $c - \frac{\epsilon}{2}$ is an upper bound for A_0 which is less than c, a contradiction. Thus we must have...

2. ...that $c \in A_0$. As $c \neq b$ and A_0 is open, there must be some $\epsilon > 0$ such that $[c, c + \epsilon) \subset A_0$. Then $c < c + \frac{\epsilon}{2} \in A_0$, so that c is not an upper bound for A_0. This contradiction forces us to conclude that $c \notin A_0 \cup B_0 = [a, b]$.

Thus c does not exist, so that there cannot be such a separation in the first place. Thus I is connected. □

Properties of Connected Spaces

Theorem 4. *Let A be a connected subspace of X, and let $A \subset B \subset \overline{A}$. Then B is connected. In particular, the closure of a connected subspace of X is also connected.*

Proof. Suppose that $B = U \cup V$ is a separation of B into nonempty disjoint open subsets. Then $A = (A \cap U) \cup (A \cap V)$ is a partition of A into disjoint open subsets, so that either $A \cap U$ or $A \cap V$ must be empty. But if $x \in U \subset \overline{A}$, we have $a \in A \cap U$, and similarly $a' \in A \cap V$. This contradicts connectedness of A. □

Theorem 5. *If A_i is a connected subspace of X for all $i \in I$, and if $p \in \cap_{i \in I} A_i$, then $\cup_{i \in I} A_i$ is connected.*

Proof. Let $A = \cup_{i \in I} A_i = U \cup V$ be a partiton of A into disjoint open subsets U and V. We show that one of the sets U or V is empty. For each i, the space $A_i = (A_i \cap U) \cup (A_i \cap V)$ a partition into disjoint open subsets. As A_i is connected, we must have one of these empty, say $A_i \cap V = \emptyset$. Then $p \in U$, so that $p \in A_i \cap U$ for all i. Thus $A_i \cap V = \emptyset$ for all i, and $V = V \cap A = V \cap (\cup A_i) = \cup (V \cap A_i) = \emptyset$. □

Theorem 6. *Let $f : X \to Y$ be continuous, and suppose that $A \subset X$ is a connected subspace. Then $f(A)$ is a connected subspace of Y.*

Proof. Suppose that $f(A) = U \cup V$, where U and V are disjoint open subsets of $f(A)$. As the map $f|_A : A \to f(A)$ is continuous by Theorem 18.2, we have that $f|_A^{-1}(U) = f^{-1}(U) \cap A$ and $f_A^{-1}(V) = f^{-1}(V) \cap A$ are open subsets of A. They are disjoint because pre-images preserve intersections. Thus one or the other is empty, say $f|_A^{-1}(V)$. Then $V = f(f^{-1}(V)) = \emptyset$. Thus $f(A)$ is connected. □

Now we can prove a result of some reknown.

Theorem 7. *(Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous. If y is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof. The image set $f([a, b])$ is connected. Suppose that y is between $f(a)$ and $f(b)$, and $y \notin f([a, b])$. Then $(-\infty, y)$ and (y, ∞) are two open sets in $\mathbb{R}$, and they form a separation of $f([a, b])$, a contradiction.

Thus $y \in f([a, b])$. □

Products of connected spaces are also connected. First, we need a lemma:

Lemma 1. *Let X and Y be homeomorphic spaces with X connected. Then Y is connected.*

Proof. Let $f : X \to Y$ be a homeomorphism, and let $Y = U \cup V$ be a partition into disjoint open sets. Then $X = f^{-1}(Y) = f^{-1}(U) \cup f^{-1}(V)$. These are disjoint open subsets of X, hence one of them, say $f^{-1}(V)$ is empty. Then $V = f(f^{-1}(V)) = \emptyset$. Thus Y is connected. □

Theorem 8. *Let X and Y be connected spaces, and let $X \times Y$ have the product topology. Then $X \times Y$ is also connected.*

Proof. Each of the slices $\{x_0\} \times Y$ and $X \times \{y_0\}$ is homeomorphic to Y and X, respectively, by a homework assignment. Thus for fixed $a \times b \in X \times Y$, the cross-shape $X \times \{b\} \cup \{a\} \times Y$ is connected, as the spaces $X \times \{b\}$ and $\{a\} \times Y$ are connected and have the point $a \times b$ in common.

The cross shapes centered at $a' \times b$ for varying $a' \in X$ and fixed $b \in Y$ all have the point $a \times b$ in common and all are connected. That they cover the space follows from $a' \times c$ belonging to the cross shape centered at $a' \times b$.

Thus from the previous theorem, their union, which is $X \times Y$, is connected.

□