Math 54 - Lecture 14: Products of Connected Spaces, Path-Connected Spaces

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July 25, 2012

Introduction

In this lecture we prove that the product of connected spaces is connected. Then we define the notion of a path-connected space. We show that path-connected spaces are connected, and exhibit some examples of path-connected spaces. We exhibit a space (the topologist's sine curve) which is connected, but not path-connected.

In the second part of the lecture we define compact spaces, and give a few examples. We show that the continuous image of a compact space is compact, and use this to prove the *Extreme Value Theorem*.

Products of Connected Spaces

Products of connected spaces are also connected. First, we need a lemma:

Lemma 1. *Let X and Y be homeomorphic spaces with X connected. Then Y is connected.*

Proof. Let $f : X \to Y$ be a homeomorphism, and let $Y = U \cup V$ be a partition into disjoint open sets. Then $X = f^{-1}(Y) = f^{-1}(U) \cup f^{-1}(V)$. These are disjoint open subsets of X, hence one of them, say $f^{-1}(V)$ is empty. Then $V = f(f^{-1}(V)) = \emptyset$. Thus Y is connected. □

Theorem 1. *Let X and Y be connected spaces, and let $X \times Y$ have the product topology. Then $X \times Y$ is also connected.*

Proof. Each of the slices $\{x_0\} \times Y$ and $X \times \{y_0\}$ is homeomorphic to Y and X, respectively, by a homework assignment. Thus for fixed $a \times b \in X \times Y$, the cross-shape $X \times \{b\} \cup \{a\} \times Y$ is connected, as the spaces $X \times \{b\}$ and $\{a\} \times Y$ are connected and have the point $a \times b$ in common.

The cross shapes centered at $a' \times b$ for varying $a' \in X$ and fixed $b \in Y$ all have the point $a \times b$ in common and all are connected. That they cover the space follows from $a' \times c$ belonging to the cross shape centered at $a' \times b$.

Thus from the previous theorem, their union, which is $X \times Y$, is connected.

□

Path-Connected Spaces

One of the nice properties of the connected spaces $\mathbb{R}^n$ is that you can construct a continuous path between any two points. Specifically, given $x, y \in \mathbb{R}^n$ the function $f : [0,1] \to \mathbb{R}^n$ defined by $f(t) = (1-t)x + ty$ is continuous with $f(0) = x$ and $f(1) = y$. We generalize this to more general spaces as follows:

Definition A topological space X is *path-connected* if for any $x, y \in X$ there exists a cts map $f : [0,1] \to X$ such that $f(0) = x$ and $f(1) = y$. We call such a map a *path* from x to y.

Lemma 2. *Any path-connected space X is connected.*

Proof. Suppose that $X = U \cup V$, where U and V are disjoint open sets. We aim to show that U or V is empty. We do this by showing that any two points of X are both in U or both in V. Let $x, y \in X$ be pts with f a path from x to y. Then $A = f([0,1])$, the image set of f, is a connected subspace of X. As $A = (A \cap U) \cup (A \cap V)$ is a decomposition of A into a disjoint union of open subsets. Thus one of these sets, say $A \cap V$ must be empty. As $x, y \in A$, we have $x, y \in U$. Thus given any two points $x, y \in X$, they are both in U or both in V. If z is any third point it must be in the same set as x and y by considering the pairs x, z and y, z. □

Lemma 3. *The continuous image of a path-connected space is path-connected.*

Proof. Let $g : X \to Y$ be cts and let X be path-connected. Suppose that $y = f(x)$ and $y' = f(x')$ are points of the image. Take a path $f : [0,1] \to X$ which connects x to x', using path-connectedness of X. Then the composition $g \circ f : [0,1] \to Y$ is a path connecting y to y'. Thus any two points in the image can be joined by a path, and the image is path-connected. □

Example Every Euclidean space is path-connected. Punctured Euclidean space $\mathbb{R}^n - \{0\}$ is path-connected if and only if $n > 1$. The map $f : \mathbb{R}^n - \{0\} \to S^{n-1}$ given by $f(x) = \frac{x}{||x||}$ is continuous and has the sphere S^{n-1} as its image. The continuous image of a path-connected space is path-connected, hence the sphere is path-connected.

Example We exhibit a space which is connected but not path-connected. Let $S = \{x \times \sin(1/x) : x \in (0,1]\}$. This is a subset of $\mathbb{R}^2$, equal to the image of $(0,1]$ under the map $f(x) = x \times \sin(1/x)$. As each of the coordinates of this function is continuous, the function is a continuous map from $(0,1]$ to $\mathbb{R}^2$. As $(0,1]$ is connected, the image S is connected. Therefore its closure $\overline{S} = S \cup \{0\} \times [-1,1]$. This space $\overline{S}$ is called the topologist's sine curve and it is NOT path-connected.

Suppose that $f : [0,1] \to \overline{S}$ is a path from 0×0 to 1×0 in the topologist's sine curve. Let $b = \sup f^{-1}(\{0\} \times [-1,1])$. Then we can take $f : [b,1] \to \overline{S}$ as a path from a point of $\{0\} \times [-1,1]$ to 1×0.

We relabel the domain as $[0,1]$ for convenience. Let $f(t) = (x(t), y(t))$ where $y(t) = \sin(1/x(t))$. As f is continuous, the coordinates x and y are continuous as well. We produce a sequence $t_n \to 0$ in $[0,1]$ such that $y(t_n)$ does not converge, which violates our sequence criterion for convergence of metric spaces.

Note that $x(1/n) > 0$ as b was selected to be the maximum element which f maps to $\{0\} \times [-1,1]$, and after relabeling b became 0. As $1/n > 0$, we must have $f(1/n) \notin \{0\} \times [-1,1]$.

Pick u such that $x(0) = 0 < u < x(1/n)$ and $\sin(1/u) = (-1)^n$. Then pick t_n such that $0 < t_n < 1/n$ and $x(t_n) = u$ by the intermediate value theorem. Then $t_n < 1/n$, so that $t_n \to 0$. But $y(t_n) = (-1)^n$, which does not converge to any number, in particular $y(0)$.

Compact Spaces

Compact spaces are a bit less intuitive than connected spaces. Some people say that compact spaces are where the "infinite and the finite kiss," an alarmingly gross image. What they mean is:

Theorem 2. *Let F be a finite subset of a topological space X. Then the following statements are true:*

1. *Every function $f : F \to \mathbb{R}$ has a minimum and a maximum value.*
2. *If $\{U_i\}$ is a collection of open sets in X, then there is some finite subcollection $U_1, \dots, U_n$ such that $F \subset U_1 \cup \dots \cup U_n$.*
3.

The truly topological generalization of finite spaces is the collection of compact spaces.

Definition Let X be a space. A collection $\mathcal{A} = \{U_i\}_{i \in I}$ of open subsets of X such that $X = \cup_i U_i$ is called an open cover of X. A subcollection $\mathcal{B} \subset \mathcal{A}$ is called a subcover if the union of the sets in $\mathcal{B}$ is also X. A subcover is finite if it contains finitely many open sets.

Example The space $[0,1]$ has an open cover $\{[0,1/2)\} \cup \{(1/n,1]\}_{n>1}$. The subcollection $\{[0,1/2),(1/3,1]\}$ is a finite subcover.

Definition Let X be a topological space. Then X is compact if every open cover has a finite subcover.

Example Any finite topological space is compact.

Example Any space equipped with the indiscrete or cofinite topology is compact.

Example A discrete space is compact if and only if it contains finitely many elements.

Theorem 3. *Let $a < b$. Then $[a,b]$ is a compact space.*

Proof. Suppose that $\mathcal{A} = \{U_i\}_{i \in I}$ is an open cover of $[a,b]$. We prove that there exists a finite subcover. Let $C = \{c \in [a,b] : [a,c]$ can be covered by a finite subcollection of $\mathcal{A}\}$. We aim to show that $b \in C$. Notice that C is bounded from above by b, as $C \subset [a,b]$. Furthermore, as we can pick a single open set U_j containing a, we have $[a,a] \subset U_j$ and hence $a \in C$. Thus C is a nonempty bounded subset of real numbers. Let x denote the least upper bound of C.

Then $x \in [a,b]$ as a is bounded by x and x is no greater than the upper bound b of C. We claim that $x = b$. Otherwise take an open set U_j which contains x. It must also contain an interval $(x-\epsilon, x+\epsilon) \subset [a,b]$. But the property of the supremum we can take an element $c \in C$ such that $x - \epsilon < c < x$. We can take a finite open subcover of $[a,c]$, say with element $U_{i_1}, \ldots, U_{i_n}$. But then adding the set U_j we obtain a finite open subcover of $[a, x+\frac{\epsilon}{2}]$. This implies that $x + \frac{\epsilon}{2} \in C$, contradicting $x = \sup C$. Thus we must have $x = b$.

The same trick we used to show that $x = b$ is used to show that $b \in C$. Take an open set U_j containing b. It must contain some half-open interval $(b-\epsilon, b] \subset U_j$. Then the property of the supremum we can take $c \in C$ such that $b - \epsilon < c < b$. But we can take a finite open subcover $U_{i_1}, \ldots, U_{i_n}$ of $[a,c]$. Then $U_j, U_{i_1}, \ldots, U_{i_n}$ is a finite open subcover of $[a,b]$. Thus $b \in C$.

So $[a,b]$ can be covered with a finite open subcover, and $[a,b]$ is compact.

□

Properties of Compact Spaces

Now that we have a good example of a compact space, we can study some of their basic properties.

Definition A subset $Y \subset X$ is called a compact subset if any open cover of Y by open sets of X has a finite subcover. Equivalently, if Y is compact in the subspace topology.

Theorem 4. *Any finite union of compact subsets of a space X is also compact.*

Proof. Just take the finite collection of all the finite open subcovers. A finite union of finite sets is finite. □

Theorem 5. *The continuous image of a compact set is compact. That is, if $f : X \to Y$ is continuous and $A \subset X$ is compact, then $f(A)$ is a compact subset of Y.*

Proof. Let $\{V_i\}_{i \in I}$ be a collection of open subsets of Y such that $f(A) \subset \cup_i V_i$. Then, given $a \in A$, there must be some j such that $f(a) \in V_j$; that is, there must be some j such that $a \in f^{-1}(V_j)$. Thus the open subsets $f^{-1}(V_j)$ form a cover of A. We can extract a finite subcover $f^{-1}(V_{j_1}), \ldots, f^{-1}(V_{j_n})$, as A is compact. Then $f(A) = f(\cup_{k=1}^n f^{-1}(V_{j_k})) \subset \cup_{k=1}^n V_{j_k}$. So $f(A)$ is compact. □