Math 54 - Lecture 15: Compact Spaces

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Compact Spaces

Definition Let X be a space. A collection $\mathcal{A} = \{U_i\}_{i \in I}$ of open subsets of X such that $X = \cup_i U_i$ is called an open cover of X. A subcollection $\mathcal{B} \subset \mathcal{A}$ is called a subcover if the union of the sets in $\mathcal{B}$ is also X. A subcover is finite if it contains finitely many open sets.

Example The space $[0,1]$ has an open cover $\{[0,1/2)\} \cup \{(1/n,1]\}_{n>1}$. The subcollection $\{[0,1/2),(1/3,1]\}$ is a finite subcover.

Definition Let X be a topological space. Then X is compact if every open cover has a finite subcover.

Example Any finite topological space is compact.

Example Any space equipped with the indiscrete or cofinite topology is compact.

Example A discrete space is compact if and only if it contains finitely many elements.

Theorem 11. *Let $a < b$. Then $[a,b]$ is a compact space.*

Proof. Suppose that $\mathcal{A} = \{U_i\}_{i \in I}$ is an open cover of $[a,b]$. We prove that there exists a finite subcover. Let $C = \{c \in [a,b] : [a,c]$ can be covered by a finite subcollection of $\mathcal{A}\}$. We aim to show that $b \in C$. Notice that C is bounded from above by b, as $C \subset [a,b]$. Furthermore, as we can pick a single open set U_j containing a, we have $[a, a+\delta] \subset U_j$ for some $\delta > 0$. Hence $a + \delta \in C$ for this δ. Thus C is a nonempty bounded subset of real numbers. Let x denote the least upper bound of C. Then $x \in [a,b]$ as a is bounded by x and x is no greater than the upper bound b of C. Note that $x \neq a$ as $a + \delta \leq x$ for the δ previously constructed. We claim that $x = b$. Otherwise take an open set U_j which contains x. It must also contain an interval $(x-\epsilon, x+\epsilon) \subset [a,b]$. But the property of the supremum we can take an element $c \in C$ such that $x - \epsilon < c < x$. We can take a finite open subcover of $[a,c]$, say with element $U_{i_1}, \ldots, U_{i_n}$. But then adding the set U_j we obtain a finite open subcover of $[a, x+\frac{\epsilon}{2}]$. This implies that $x + \frac{\epsilon}{2} \in C$, contradicting $x = \sup C$. Thus we must have $x = b$.

The same trick we used to show that $x = b$ is used to show that $b \in C$. Take an open set U_j containing b. It must contain some half-open interval $(b-\epsilon, b] \subset U_j$. Then the property of the supremum we can take $c \in C$ such that $b - \epsilon < c < b$. But we can take a finite open subcover $U_{i_1}, \ldots, U_{i_n}$ of $[a,c]$. Then $U_j, U_{i_1}, \ldots, U_{i_n}$ is a finite open subcover of $[a,b]$. Thus $b \in C$.

So $[a, b]$ can be covered with a finite open subcover, and $[a, b]$ is compact.

□

Properties of Compact Spaces

Now that we have a good example of a compact space, we can study some of their basic properties.

Definition A subset $Y \subset X$ is called a compact subset if any open cover of Y by open sets of X has a finite subcover. Equivalently, if Y is compact in the subspace topology.

Theorem 12. *Any finite union of compact subsets of a space X is also compact.*

Proof. Just take the finite collection of all the finite open subcovers. A finite union of finite sets is finite. □

Theorem 13. *The continuous image of a compact set is compact. That is, if $f : X \to Y$ is continuous and $A \subset X$ is compact, then $f(A)$ is a compact subset of Y.*

Proof. Let $\{V_i\}_{i\in I}$ be a collection of open subsets of Y such that $f(A) \subset \cup_i V_i$. Then, given $a \in A$, there must be some j such that $f(a) \in V_j$; that is, there must be some j such that $a \in f^{-1}(V_j)$. Thus the open subsets $f^{-1}(V_j)$ form a cover of A. We can extract a finite subcover $f^{-1}(V_{j_1}), \ldots, f^{-1}(V_{j_n})$, as A is compact. Then $f(A) = f(\cup_{k=1}^n f^{-1}(V_{j_k})) \subset \cup_{k=1}^n V_{j_k}$. So $f(A)$ is compact. □

Theorem 14. *Every closed subset of a compact space X is compact.*

Proof. Suppose that C is a closed subset of X. Let $\mathcal{A} = \{U_i\}_{i\in I}$ be a collection of open subsets of X whose union contains C. We produce a finite sub-collection whose union still contains C. Consider the collection $\mathcal{A}' = \{U_i\}_i \cup \{X - C\}$. As C is closed, this is a collection of open subsets of X. The union contains $\cup_i U_i$, hence it contains C, and it contains $X - C$. Thus $\mathcal{A}'$ is an open covering of X. Let $\{U_{i_k}\}_{k=1}^n \cup \{X - C\}$ be a finite sub-cover contained in $\mathcal{A}'$. Then $\{U_{i_k}\}_{k=1}^n$ is a sub-collection of $\mathcal{A}$ such that the union of all elements in $\mathcal{A}$ contains A. □

Theorem 15. *Every compact subset of a Hausdorff space is closed.*

Proof. Let $K \subset X$ be a compact subset of the Hausdorff space X. We show that K is closed. We do this by showing that the complement $X - K$ is an open subset, via the point-wise formulation of openness from the second homework assignment. Let $x \in X - K$. We produce a nbhd U st $x \in U \subset X - K$. For any $y \in K$, we pick disjoint nbhds U_y and V_y of x and y respectively. Note that $K \subset \cup_{y\in K} V_y$. As $\{V_y\}_{y\in K}$ is an open cover of K, we can select some finite subcover $V_{y_1}, \ldots, V_{y_n}$. Then $x \in U_{y_1}, \ldots, U_{y_n}$ by construction. Then $x \in \cap_{i=1}^n U_{y_i}$, and the latter set, which we call U, is open. We need to show that this nbhd

But for any $z \in K$, there is a V_{y_i} such that $z \in V_{y_i}$. But then $z \notin U_{y_i}$, so that $z \notin U$. Thus U contains no elements of K, and $U \subset X - K$. □

The following theorem is remarkably useful when dealing with compact Hausdorff spaces.

Theorem 16. *Let $f : X \to Y$ be a continuous map from a compact space X to a Hausdorff space. Then f is closed; that is, if $C \subset X$ is a closed set, then $f(C) \subset Y$ is closed.*

Proof. If C is closed in X, then C is compact. The image of C is therefore a compact subset of Y. As Y is compact, any compact subset of Y is closed. Thus $f(C)$ is closed. So f is a closed map. □

Product Spaces

Just as the product of connected spaces is connected, the product of compact spaces is compact.

Theorem 17. *If X and Y are compact spaces, then $X \times Y$ is compact.*

In order to prove this, it helps to distinguish a valuable lemma.

Lemma 6. *(The Tube Lemma) Let X and Y be spaces, with Y compact. If $N \subset X \times Y$ is open, and a slice $\{x_0\} \times Y \subset N$ for some $x_0 \in X$, then there is a nbhd W of x_0 such that $W \times Y \subset N$.*

Proof. For each y, consider a basis element $U_y \times V_y$ of $X \times Y$ such that $\{x_0\} \times y \in U_y \times V_y \subset N$. (We can do this as N contains all the points of the slice $\{x_0\} \times Y$ and is an open set). Note that $y \in V_y$ and $x_0 \in U_y$ for all y. Then the open sets $\{V_y\}_{y \in Y}$ form an open cover of the factor space Y, because if $y \in Y$, then $\{x_0\} \times y \in U_y \times V_y$. As Y is compact, we can select a finite subcover $V_{y_1}, \ldots, V_{y_n}$ such that $Y = \cup_{i=1}^{n} V_{y_i}$. Let $W = \cap_{i=1}^{n} U_{y_i}$, a nbhd of x_0. Then W is the desired nbhd of x_0. Suppose that $(z, y) \in W \times Y$. Then there is an j such that $y \in V_{y_j}$, as the collection $\{V_{y_i}\}_{i=1}^{n}$ covers Y. As $z \in W = \cap_{i=1}^{n} U_{y_i}$, we have $(z, y) \in U_j \times V_j \subset N$. □

Proof. (Theorem).

Let $\{P_i\}_{i \in I}$ be a collection of open subsets of $X \times Y$ covering $X \times Y$. For each $x \in X$, we can cover the space $\{x\} \times Y$ with finitely many open sets, say P_k^x for $k = 1, \ldots, n_x$ (the x indicates that these open sets are intended to cover the slice $\{x\} \times Y$). By the tube lemma we can take an open set W^x such that $x \in W^x$ and $W^x \times Y \subset Q^x = \cup_{k=1}^{n_x} P_k^x$. Then the open sets $\{W^x\}$ cover the compact space X. Thus we can take a finite subcover $W^{x_1}, \ldots, W^{x_m}$. Claim: The collection of all $P_k^{x_j}$ for $k = 1, \ldots, n_{x_j}$ and $j = 1, \ldots, m$, form a subcover of $\{P_i\}_{i \in I}$. For if $x \times y \in X \times Y$, we can pick W^{x_j} such that $x \in W^{x_j}$. Then $W^{x_j} \times Y \subset \cup_{k=1}^{n_{x_j}} P_k^{x_j}$, so that there exists k between 1 and n_{x_j} such that $x \times y \in P_k^{x_j}$. Thus we have constructed the finite subcover of $X \times Y$, so that the latter set is compact. □

Corollary 1. *Any finite product of compact spaces is compact.*

Proof. If $X_1, \ldots, X_n$ is a collection of spaces, then $(X_1 \times \ldots \times X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times \ldots \times X_n$ via the map $((x_1, \ldots, x_{n-1}), x_n) \to (x_1, \ldots, x_n)$. Then induction and the fact that the continuous image of a compact space is compact finish the proof. □

Corollary 2. *Any product of closed intervals in* $\mathbb{R}$*, say* $\prod_{i=1}^{n}[a_i, b_i]$*, is compact in the product topology.*

Theorem 18. *(Tychonoff's Theorem) If* $\{X_i\}_{i\in I}$ *is a collection of compact spaces, then* $\prod_i X_i$ *is compact in the product topology.*

Proof. You can read about it in chapter 5. Interesting: Tychonoff's theorem is equivalent to the axiom of choice. □

The Finite Intersection Property and Cantor's Intersection Theorem

We can restate the definition of compactness as follows: **Definition** A collection $\mathcal{C}$ of subsets of a set X has the finite intersection property if every finite subcollection $\{C_1, \ldots, C_n\} \subset \mathcal{C}$ has non-empty intersection $C_1 \cap \ldots \cap C_n$.

Theorem 19. *Let* X *be a space. Then* X *is compact if and only if every collection* $\mathcal{C}$ *of closed subsets of* X *with the FIP has nonempty intersection* $\cap_{C\in\mathcal{C}} C$.

Proof. This is just using the contrapositive and DeMorgan's laws. Read about it in Munkres if you need the details. □

We can use the language of the finite intersection property to prove a famous result of Cantor's called the Intersection Theorem.

Theorem 20. *Let* $C_1 \supset C_2 \ldots$ *be a descending chain of non-empty closed subsets of the compact space* X*. Then* $\cap_{n=1}^{\infty} C_n \neq \emptyset$.

(We will use this theorem later to give a purely topological proof that the reals are uncountable).

Proof. Note that as X is compact, X must have the finite intersection property. If you take any finite collection $C_{n_1}, \ldots, C_{n_k}$ their intersection is simply C_n, where $n = \max n_i$. This is non-empty, so the intersection of any finite subcollection of $\{C_n\}_{n=1}^{\infty}$ is non-empty. Thus the finite intersection property implies that $\cap_{n=1}^{\infty} C_n$ is non-empty. □