Compact Spaces

**Definition** Let $X$ be a space. A collection $\mathcal{A} = \{U_i\}_{i \in I}$ of open subsets of $X$ such that $X = \bigcup_{i \in I} U_i$ is called an open cover of $X$. A subcollection $\mathcal{B} \subset \mathcal{A}$ is called a subcover if the union of the sets in $\mathcal{B}$ is also $X$. A subcover is finite if it contains finitely many open sets.

**Example** The space $[0, 1]$ has an open cover $\{[0, 1/2)\} \cup \{(1/n, 1]\}$ for $n > 1$. The subcollection $\{[0, 1/2), (1/3, 1]\}$ is a finite subcover.

**Definition** Let $X$ be a topological space. Then $X$ is compact if every open cover has a finite subcover.

**Example** Any finite topological space is compact.

**Example** Any space equipped with the indiscrete or cofinite topology is compact.

**Example** A discrete space is compact if and only if it contains finitely many elements.

**Theorem 11.** Let $a < b$. Then $[a, b]$ is a compact space.

**Proof.** Suppose that $\mathcal{A} = \{U_i\}_{i \in I}$ is an open cover of $[a, b]$. We prove that there exists a finite subcover. Let $C = \{c \in [a, b] : [a, c] \text{ can be covered by a finite subcollection of } \mathcal{A}\}$. We aim to show that $b \in C$. Notice that $C$ is bounded from above by $b$, as $C \subset [a, b]$. Furthermore, as we can pick a single open set $U_j$ containing $a$, we have $[a, a + \delta] \subset U_j$ for some $\delta > 0$. Hence $a + \delta \in C$ for this $\delta$. Thus $C$ is a nonempty bounded subset of real numbers. Let $x$ denote the least upper bound of $C$. Then $x \in [a, b]$ as $a$ is bounded by $x$ and $x$ is no greater than the upper bound $b$ of $C$. Note that $x \neq a$ as $a + \delta \leq x$ for the $\delta$ previously constructed. We claim that $x = b$. Otherwise take an open set $U_j$ which contains $x$. It must also contain an interval $(x - \epsilon, x + \epsilon) \subset [a, b]$. But the property of the supremum we can take an element $c \in C$ such that $x - \epsilon < c < x$. We can take a finite open subcover of $[a, c]$, say with element $U_{i_1}, \ldots, U_{i_n}$. But then adding the set $U_j$ we obtain a finite open subcover of $[a, x + \epsilon]$. This implies that $x + \frac{\epsilon}{2} \in C$, contradicting $x = \sup C$. Thus we must have $x = b$.

The same trick we used to show that $x = b$ is used to show that $b \in C$. Take an open set $U_j$ containing $b$. It must contain some half-open interval $(b - \epsilon, b] \subset U_j$. Then the property of the supremum we can take $c \in C$ such that $b - \epsilon < c < b$. But we can take a finite open subcover $U_{i_1}, \ldots, U_{i_n}$ of $[a, c]$. Then $U_j, U_{i_1}, \ldots, U_{i_n}$ is a finite open subcover of $[a, b]$. Thus $b \in C$. 1
Thus subsets of \( X \) contains \( C \). The latter set, which we call \( \{ V_i \}_{i \in I} \) form a cover of \( A \). We can extract a finite subcover \( f^{-1}(V_j), \ldots, f^{-1}(V_n) \), as \( A \) is compact. Then \( f(A) = f(\bigcup_{k=1}^{n} f^{-1}(V_{j_k})) \subset \bigcup_{k=1}^{n} V_{j_k} \). So \( f(A) \) is compact.

**Theorem 14.** Every closed subset of a compact space \( X \) is compact.

**Proof.** Suppose that \( C \) is a closed subset of \( X \). Let \( \mathcal{A} = \{ U_i \}_{i \in I} \) be a collection of open subsets of \( X \) whose union contains \( C \). We produce a finite sub-collection whose union still contains \( C \). Consider the collection \( \mathcal{A}' = \{ U_i \}_{i \in I} \cup \{ X - C \} \). As \( C \) is closed, this is a collection of open subsets of \( X \). The union contains \( \bigcup_i U_i \), hence it contains \( C \), and it contains \( X - C \). Thus \( \mathcal{A}' \) is an open covering of \( X \). Let \( \{ U_{i_k} \}_{k=1}^{n} \cup \{ X - C \} \) be a finite sub-cover contained in \( \mathcal{A}' \). Then \( \{ U_{i_k} \}_{k=1}^{n} \) is a sub-collection of \( \mathcal{A} \) such that the union of all elements in \( \mathcal{A} \) contains \( C \).

**Theorem 15.** Every compact subset of a Hausdorff space \( X \) is closed.

**Proof.** Let \( K \subset X \) be a compact subset of the Hausdorff space \( X \). We show that \( K \) is closed. We do this by showing that the complement \( X - K \) is an open subset, via the point-wise formulation of openness from the second homework assignment. Let \( x \in X - K \). We produce a nbhd \( U \) of \( x \). For any \( y \in K \), we pick disjoint nbhds \( U_y \) and \( V_y \) of \( x \) and \( y \) respectively. Note that \( K \subset \bigcup_{y \in K} V_y \). As \( \{ V_y \}_{y \in K} \) is an open cover of \( K \), we can select some finite subcover \( V_{y_1}, \ldots, V_{y_n} \). Then \( x \in U_{y_1}, \ldots, U_{y_n} \) by construction. Then \( x \in \bigcap_{i=1}^{n} U_{y_i} \), and the latter set, which we call \( U \), is open. We need to show that this nbhd.

But for any \( z \in K \), there is a \( V_{y_i} \) such that \( z \in V_{y_i} \). But then \( z \not\in U_{y_i} \), so that \( z \not\in U \). Thus \( U \) contains no elements of \( K \), and \( U \subset X - K \).
The following theorem is remarkably useful when dealing with compact Hausdorff spaces.

**Theorem 16.** Let \( f : X \to Y \) be a continuous map from a compact space \( X \) to a Hausdorff space. Then \( f \) is closed; that is, if \( C \subseteq X \) is a closed set, then \( f(C) \subseteq Y \) is closed.

**Proof.** If \( C \) is closed in \( X \), then \( C \) is compact. The image of \( C \) is therefore a compact subset of \( Y \). As \( Y \) is compact, any compact subset of \( Y \) is closed. Thus \( f(C) \) is closed. So \( f \) is a closed map. \( \square \)

**Product Spaces**

Just as the product of connected spaces is connected, the product of compact spaces is compact.

**Theorem 17.** If \( X \) and \( Y \) are compact spaces, then \( X \times Y \) is compact.

In order to prove this, it helps to distinguish a valuable lemma.

**Lemma 6.** (The Tube Lemma) Let \( X \) and \( Y \) be spaces, with \( Y \) compact. If \( N \subseteq X \times Y \) is open, and a slice \( \{ x_0 \} \times Y \subseteq N \) for some \( x_0 \in X \), then there is a nbhd \( W \) of \( x_0 \) such that \( W \times Y \subseteq N \).

**Proof.** For each \( y \), consider a basis element \( U_y \times V_y \) of \( X \times Y \) such that \( \{ x_0 \} \times y \subseteq U_y \times V_y \subseteq N \). (We can do this as \( N \) contains all the points of the slice \( \{ x_0 \} \times Y \) and is an open set.) Note that \( y \in V_y \) and \( x_0 \in U_y \) for all \( y \). Then the open sets \( \{ V_y \}_{y \in Y} \) form an open cover of the factor space \( Y \), because if \( y \in Y \), then \( \{ x_0 \} \times y \subseteq U_y \times V_y \). As \( Y \) is compact, we can select a finite subcover \( V_{y_1}, \ldots, V_{y_n} \) such that \( Y = \bigcup_{i=1}^n V_{y_i} \). Let \( W = \bigcap_{i=1}^n U_{y_i} \), a nbhd of \( x_0 \). Then \( W \) is the desired nbhd of \( x_0 \). Suppose that \((z, y) \in W \times Y \). Then there is an \( j \) such that \( y \in V_{y_j} \), as the collection \( \{ V_{y_i} \}_{i=1}^n \) covers \( Y \). As \( z \in W = \bigcap_{i=1}^n U_{y_i} \), we have \((z, y) \in U_j \times V_j \subseteq N \). \( \square \)

**Proof.** (Theorem)

Let \( \{ P_i \}_{i \in I} \) be a collection of open subsets of \( X \times Y \) covering \( X \times Y \). For each \( x \in X \), we can cover the space \( \{ x \} \times Y \) with finitely many open sets, say \( P_x^k \) for \( k = 1, \ldots, n_x \) (the \( x \) indicates that these open sets are intended to cover the slice \( \{ x \} \times Y \)). By the tube lemma we can take an open set \( W^x \) such that \( x \in W^x \) and \( W^x \times Y \subseteq Q^x = \bigcup_{k=1}^{n_x} P_x^k \). Then the open sets \( \{ W^x \} \) cover the compact space \( X \). Thus we can take a finite subcover \( W^{x_1}, \ldots, W^{x_m} \).

Claim: The collection of all \( P_k^{x_j} \) for \( k = 1, \ldots, n_{x_j} \) and \( j = 1, \ldots, m \), form a subcover of \( \{ P_i \}_{i \in I} \). For if \( x \times y \in X \times Y \), we can pick \( W^{x_j} \) such that \( x \in W^{x_j} \). Then \( W^{x_j} \times Y \subseteq \bigcup_{k=1}^{n_{x_j}} P_k^{x_j} \), so that there exists \( k \) between 1 and \( n_{x_j} \) such that \( x \times y \in P_k^{x_j} \). Thus we have constructed the finite subcover of \( X \times Y \), so that the latter set is compact. \( \square \)

**Corollary 1.** Any finite product of compact spaces is compact.

**Proof.** If \( X_1, \ldots, X_n \) is a collection of spaces, then \((X_1 \times \ldots \times X_{n-1}) \times X_n\) is homeomorphic to \( X_1 \times \ldots \times X_n \) via the map \((x_1, \ldots, x_{n-1}, x_n) \to (x_1, \ldots, x_n)\). Then induction and the fact that the continuous image of a compact space is compact finish the proof. \( \square \)
Corollary 2. Any product of closed intervals in $\mathbb{R}$, say $\prod_{i=1}^{n}[a_i, b_i]$, is compact in the product topology.

Theorem 18. (Tychonoff’s Theorem) If $\{X_i\}_{i \in I}$ is a collection of compact spaces, then $\prod_{i} X_i$ is compact in the product topology.

Proof. You can read about it in chapter 5. Interesting: Tychonoff’s theorem is equivalent to the axiom of choice. □

The Finite Intersection Property and Cantor’s Intersection Theorem

We can restate the definition of compactness as follows: Definition A collection $C$ of subsets of a set $X$ has the finite intersection property if every finite subcollection $\{C_1, \ldots, C_n\} \subset C$ has non-empty intersection $C_1 \cap \ldots \cap C_n$.

Theorem 19. Let $X$ be a space. Then $X$ is compact if and only if every collection $C$ of closed subsets of $X$ with the FIP has nonempty intersection $\cap_{C \in C} C$.

Proof. This is just using the contrapositive and DeMorgan’s laws. Read about it in Munkres if you need the details. □

We can use the language of the finite intersection property to prove a famous result of Cantor’s called the Intersection Theorem.

Theorem 20. Let $C_1 \supset C_2 \ldots$ be a descending chain of non-empty closed subsets of the compact space $X$. Then $\cap_{n=1}^{\infty} C_n \neq \emptyset$.

(We will use this theorem later to give a purely topological proof that the reals are uncountable).

Proof. Note that as $X$ is compact, $X$ must have the finite intersection property. If you take any finite collection $C_{n_1}, \ldots, C_{n_k}$ their intersection is simply $C_n$, where $n = \max n_i$. This is non-empty, so the intersection of any finite subcollection of $\{C_n\}_{n=1}^{\infty}$ is non-empty. Thus the finite intersection property implies that $\cap_{n=1}^{\infty} C_n$ is non-empty. □