Math 54 - Lecture 16: Compact Hausdorff Spaces, Products of Compact Spaces

Dan Crytser

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Introduction

In this lecture, we review some of the properties of compact spaces we saw in the last lecture. Then we show that every closed subset of a compact space is compact, and show that the converse holds provided the space is Hausdorff. To do this, we show that compact Hausdorff spaces are important examples of *regular spaces*, an example of a separation axiom. Then we prove an important minor result called the *Tube Lemma* and use it to prove that the finite product of compact spaces is again compact. The extension of this to arbitrary products of families of compact spaces is true, a famous result called *Tychonoff's Theorem*, which we will not prove. Then we note an equivalent formulation of compactness, called the Finite Intersection Property (FIP), and use it as a convenient way to prove *Cantor's Intersection Theorem*. We will use a version of this theorem to prove that the real numbers are not countable in a future lecture.

Properties of Compact Spaces, ctd.

Theorem 1. *Every closed subset of a compact space X is compact.*

Proof. Suppose that C is a closed subset of X. Let $\mathcal{A} = \{U_i\}_{i \in I}$ be a collection of open subsets of X whose union contains C. We produce a finite sub-collection whose union still contains C. Consider the collection $\mathcal{A}' = \{U_i\}_i \cup \{X - C\}$. As C is closed, this is a collection of open subsets of X. The union contains $\cup_i U_i$, hence it contains C, and it contains $X - C$. Thus $\mathcal{A}'$ is an open covering of X. Let $\{U_{i_k}\}_{k=1}^n \cup \{X - C\}$ be a finite sub-cover contained in $\mathcal{A}'$. Then $\{U_{i_k}\}_{k=1}^n$ is a sub-collection of $\mathcal{A}$ such that the union of all elements in $\mathcal{A}$ contains A. □

Theorem 2. *Every compact subset of a Hausdorff space is closed.*

Proof. Let $K \subset X$ be a compact subset of the Hausdorff space X. We show that K is closed. We do this by showing that the complement $X - K$ is an open subset, via the point-wise formulation of openness from the second homework assignment. Let $x \in X - K$. We produce a nbhd U st $x \in U \subset X - K$. For any $y \in K$, we pick disjoint nbhds U_y and V_y of x and y respectively. Note that $K \subset \cup_{y \in K} V_y$. As $\{V_y\}_{y \in K}$ is an open cover of K, we can select some finite subcover $V_{y_1}, \ldots, V_{y_n}$. Then $x \in U_{y_1}, \ldots, U_{y_n}$ by construction. Then $x \in \cap_{i=1}^n U_{y_i}$, and the latter set, which we call U, is open. We need to show that this nbhd

But for any $z \in K$, there is a V_{y_i} such that $z \in V_{y_i}$. But then $z \notin U_{y_i}$, so that $z \notin U$. Thus U contains no elements of K, and $U \subset X - K$. □

The following theorem is remarkably useful when dealing with compact Hausdorff spaces.

Theorem 3. *Let $f : X \to Y$ be a continuous map from a compact space X to a Hausdorff space. Then f is closed; that is, if $C \subset X$ is a closed set, then $f(C) \subset Y$ is closed.*

Proof. If C is closed in X, then C is compact. The image of C is therefore a compact subset of Y. As Y is compact, any compact subset of Y is closed. Thus $f(C)$ is closed. So f is a closed map. □

Product Spaces

Just as the product of connected spaces is connected, the product of compact spaces is compact.

Theorem 4. *If X and Y are compact spaces, then $X \times Y$ is compact.*

In order to prove this, it helps to distinguish a valuable lemma.

Lemma 1. *(The Tube Lemma) Let X and Y be spaces, with Y compact. If $N \subset X \times Y$ is open, and a slice $\{x_0\} \times Y \subset N$ for some $x_0 \in X$, then there is a nbhd W of x_0 such that $W \times Y \subset N$.*

Proof. For each y, consider a basis element $U_y \times V_y$ of $X \times Y$ such that $\{x_0\} \times y \in U_y \times V_y \subset N$. (We can do this as N contains all the points of the slice $\{x_0\} \times Y$ and is an open set). Note that $y \in V_y$ and $x_0 \in U_y$ for all y. Then the open sets $\{V_y\}_{y \in Y}$ form an open cover of the factor space Y, because if $y \in Y$, then $\{x_0\} \times y \in U_y \times V_y$. As Y is compact, we can select a finite subcover $V_{y_1}, \ldots, V_{y_n}$ such that $Y = \cup_{i=1}^n V_{y_i}$. Let $W = \cap_{i=1}^n U_{y_i}$, a nbhd of x_0. Then W is the desired nbhd of x_0. Suppose that $(z, y) \in W \times Y$. Then there is an j such that $y \in V_{y_j}$, as the collection $\{V_{y_i}\}_{i=1}^n$ covers Y. As $z \in W = \cap_{i=1}^n U_{y_i}$, we have $(z, y) \in U_j \times V_j \subset N$. □

Proof. (Theorem).

Let $\{P_i\}_{i \in I}$ be a collection of open subsets of $X \times Y$ covering $X \times Y$. For each $x \in X$, we can cover the space $\{x\} \times Y$ with finitely many open sets, say P_k^x for $k = 1, \ldots, n_x$ (the x indicates that these open sets are intended to cover the slice $\{x\} \times Y$). By the tube lemma we can take an open set W^x such that $x \in W^x$ and $W^x \times Y \subset Q^x = \cup_{k=1}^{n_x} P_k^x$. Then the open sets $\{W^x\}$ cover the compact space X. Thus we can take a finite subcover $W^{x_1}, \ldots, W^{x_m}$. Claim: The collection of all $P_k^{x_j}$ for $k = 1, \ldots, n_{x_j}$ and $j = 1, \ldots, m$, form a subcover of

$\{P_i\}_{i\in I}$. For if $x \times y \in X \times Y$, we can pick W^{x_j} such that $x \in W^{x_j}$. Then $W^{x_j} \times Y \subset \cup_{k=1}^{n_{x_j}} P_k^{x_j}$, so that there exists k between 1 and n_{x_j} such that $x \times y \in P_k^{x_j}$. Thus we have constructed the finite subcover of $X \times Y$, so that the latter set is compact. □

Corollary 1. *Any finite product of compact spaces is compact.*

Proof. If $X_1, \ldots, X_n$ is a collection of spaces, then $(X_1 \times \ldots \times X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times \ldots \times X_n$ via the map $((x_1, \ldots, x_{n-1}), x_n) \to (x_1, \ldots, x_n)$. Then induction and the fact that the continuous image of a compact space is compact finish the proof. □

Corollary 2. *Any product of closed intervals in $\mathbb{R}$, say $\prod_{i=1}^{n}[a_i, b_i]$, is compact in the product topology.*

Theorem 5. *(Tychonoff's Theorem) If $\{X_i\}_{i\in I}$ is a collection of compact spaces, then $\prod_i X_i$ is compact in the product topology.*

Proof. You can read about it in chapter 5. Interesting: Tychonoff's theorem is equivalent to the axiom of choice. □

The Finite Intersection Property and Cantor's Intersection Theorem

Definition A collection $\mathcal{C}$ of subsets of X has the finite intersection property if every finite subcollection $\{C_1, \ldots, C_n\} \subset \mathcal{C}$ has nonempty intersection, i.e. $C_1 \cap \ldots \cap C_n \neq \emptyset$.

Theorem 6. *Let X be a space. Then X is compact if and only if every collection $\mathcal{C}$ of subsets with FIP has nonempty intersection $\cap_{C\in\mathcal{C}} C \neq \emptyset$.*

Proof. This is just using the contrapositive and DeMorgan's laws. Read about it in Munkres if you need the details. □

We can use the language of the finite intersection property to prove a famous result of Cantor's called the Intersection Theorem.

Theorem 7. *Let $C_1 \supset C_2 \ldots$ be a descending chain of non-empty closed subsets of the compact space X. Then $\cap_{n=1}^{\infty} C_n \neq \emptyset$.*

(We will use this theorem later to give a purely topological proof that the reals are uncountable).

Proof. Note that as X is compact, X must have the finite intersection property. If you take any finite collection $C_{n_1}, \ldots, C_{n_k}$ their intersection is simply C_n, where $n = \max n_i$. This is non-empty, so the intersection of any finite subcollection of $\{C_n\}_{n=1}^{\infty}$ is non-empty. Thus the finite intersection property implies that $\cap_{n=1}^{\infty} C_n$ is non-empty. □