Math 54 - Lecture 17: Cantor's Theorem and Locally Compact Spaces

Dan Crytser

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Introduction

In this section we prove Cantor's Intersection Theorem, which is used later to prove the uncountability of the real number line. Then we examine locally compact spaces, which are a flexible generalization of compact spaces. We show that you can embed a locally compact Hausdorff space into a compact Hausdorff space as a dense open subset. This process is known as *one-point compactification*.

Cantor's Intersection Theorem

Cantor's theorem states that, in compact spaces, the intersection of a nested chain of closed subsets $C_1 \supset C_2 \supset \ldots$ is non-empty. This fails for non-compact spaces. For example $A_n = [n, \infty)$ forms such a chain of closed subsets of $\mathbb{R}$, yet the intersection in $\mathbb{R}$ is $\emptyset$.

Theorem 1. *Let X be a compact space, let $C_1 \supset C_2 \ldots$ be a nested chain of non-empty closed subsets. Then $\cap C_n \neq \emptyset$.*

Proof. Suppose that $\cap C_n = \emptyset$ and let $U_n = X - C_n$ for all n. Note that $\{U_n\}$ is an open cover of X, as $X = X - \cap C_n = \cup X - C_n = \cup U_n$. Extract a finite subcover. As $U_1 \subset U_2 \ldots$, one of the U_k must equal X. Then $C_k = X - U_k = \emptyset$, contradicting non-emptiness of all the C_k. Thus the intersection is non-empty. $\square$

Compact Hausdorff Spaces

Now we give a result which implies that certain compact Hausdorff spaces are uncountable. When you study the Cantor Set in this week's homework, you can use this result to give an elegant proof of that set's uncountability.

Definition A pt $x \in X$ is isolated if $\{x\}$ is open.

Theorem 2. *Let X be a non-empty compact Hausdorff space with no isolated points. Then X is uncountable.*

Proof. Claim: given $U \subset X$ non-empty and open and $x \in X$, we can construct $V \subset U$ open such that $x \notin \overline{V}$. Choose a point y of U different from x (any pt of U if $x \notin U$ and any point of $U - \{x\}$ else–remember x is not isolated). Take disjoint nbhds W_1 and W_2 of x and y. Let $V = W_2 \cap U$. Then $x \notin \overline{V}$ as W_1 is a nbhd of x that does not contain any points of V.

Now, given any sequence $x_1, x_2, \ldots$ of points of X, we produce x not in the sequence. Take a nonempty open set V_1 not containing x_1 by the first part, letting $U = X$. We construct a sequence $\overline{V_1} \supset \overline{V_2} \ldots$ of non-empty closed subsets, by the rule: if V_n is constructed, take a nbhd V_{n+1} contained in V_n such that $\overline{V_{n+1}}$ does not include x_{n+1}. Then the inclusion of closures follows from $A \subset B \rightarrow \overline{A} \subset \overline{B}$. The intersection $W = \cap_n \overline{V_n}$ is non-empty by compactness of X and Cantor's Intersection theorem. Yet $x_n \notin \overline{V_n}$, hence no point in the sequence is contained in the non-empty set W.

□

Example As the space $[0, 1]$ is compact and Hausdorff without any isolated points, it is uncountable. Thus the real number line is uncountable as well.

Local Compactness

Definition A Hausdorff space X is called locally compact if every point x has a nbhd U such that $\overline{U}$ is compact.

Example The real line is not compact. However it is locally compact, as $\overline{(a, b)} = [a, b]$ a compact set for any basis element.

Example Any discrete space is locally compact. For the singleton $\{x\}$ is a nbhd which equals its closure, and its closure is compact. A discrete space is compact if and only if it is finite, which you can check by considering the open cover by singletons.

Example Any compact space is locally compact by taking the nbhd to be the total set X.

Compactness in Euclidean Space

One of the easiest cases in which you can determine all the compact subsets of a topological space is the Euclidean spaces $\mathbb{R}^n$. **Definition** A subset Y of a metric space X is bounded if there exists $C > 0$ such that $d(x, y) < C$ for all $x, y \in Y$.

Theorem 3. *(Heine-Borel Theorem) Let $Y \subset \mathbb{R}^n$. Then Y is compact if and only if Y is bounded in the square metric (equiv, the Euclidean metric) and closed.*

Proof. Suppose that Y is compact. The open balls $B(0, n)$ for $n \in \mathbb{N}$ form an open cover of Y, so one of these must contain Y by compactness and nestedness (a word?!). Then Y

is bounded by the triangle inequality. As $\mathbb{R}^n$ is Hausdorff, all compact subsets are closed. Therefore Y is closed and bounded.

Suppose that Y is closed and bounded in the square metric. Case 1: $0 \in Y$. Suppose that $d(y, y') < C$ for all $y, y' \in Y$, where $C > 0$. Then $Y \subset [-C, C]^n$ and as Y is a closed subset of a compact set it is compact.

Case 2: general. Let $y \in Y$. Then set $D = C + d(x, 0)$. Then $Y \subset [-D, D]^n$ and the proof goes through as before.

$\square$

Theorem 4. *Euclidean space $\mathbb{R}^n$ is locally compact for all n.*

Proof. Any open ball $B(x, \epsilon)$ has compact closure $\overline{B(x, \epsilon)} = \{y \in \mathbb{R}^n : d(x, y) \leq \epsilon\}$. $\square$