# Math 54 - Lecture 18: Countability Axioms

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## Introduction

The next two weeks will be given to the study of two purely topological notions: countability and separation axioms. Roughly speaking, a countability axiom says when you can describe all the open sets of a topological space, or all the nbhds of a fixed pt, using a countable basis. A separation axiom tells you when you can separate two closed subsets with nbhds, as in the Hausdorff condition (where you might put conditions on one or more of the closed subsets).

## First-Countable Spaces

**Definition** Let $X$ be a space and $x \in X$. Then a *countable basis at* $x$ is a sequence $U_1, \ldots$ of nbhds of $x$ such that if $U$ is a nbhd of $x$, then $U_n \subset U$ for some $n$. The sets $U_n$ need not be distinct.

Note that a countable basis at $x$ does not need to include every nbhd of $x$ in the sequence.

**Example** Let $X = \mathbb{R}$ and $x = 0$ in the standard top. The sequence $\{(-1/n, 1/n)\}_{n \geq 1}$ is a countable basis at 0, for if $(a, b)$ contains 0, then taking $n$ such that $1/n < |a| + |b|$ implies that $(-1/n, 1/n) \subset (a, b)$.

**Definition** A space $X$ is *first-countable* if it has a countable basis at each point $x \in X$.

The primary useful fact about first-countable spaces is that they have the same behavior as metric spaces regarding the sequence characterization of closures and continuity.

**Theorem 22.** *Let $X$ be a space.*

*(a) If $A \subset X$ and $x \in X$ and if there exists a sequence $(a_n) \subset A$ converging to $x$, then $x \in \overline{A}$. The converse holds for $X$ first-countable.*

*(b) Let $f : X \to Y$ be a function into a space $Y$. If $f$ is continuous, then for every sequence $x_n$ converging to $x$ in $X$, the sequence $f(x_n) \to f(x)$ in $Y$. The converse holds for $X$ first-countable.*

The proofs are the same as in the metric case, you just use take a countable basis at $x$, say $U_1, \ldots$ and use $U_n$ to replace the behavior of $B(x, \frac{1}{n})$.

# Second-Countable Spaces

Even nicer than first-countable spaces are the second-countable, in which we can describe all the open sets in terms of a countable subcollection.

**Definition** A space $X$ is second-countable if it has a countable basis for its topology, say $U_1, \ldots$. That is, given any open set $U$ and point $x \in U$, there is $U_n \subset U$ with $x \in U_n$.

Note: any second-countable space is first-countable, as we can take for a basis at $x \in X$ the sequence of all $U_n$ which contain $x$.

**Example** Any $\mathbb{R}^n$ is second-countable, as the products $(a_1, b_1) \times \ldots \times (a_n, b_n)$ with each $a_i, b_i$ rational form a countable basis.

**Example** The box topology on $\mathbb{R}^\omega$ is not first-countable. For suppose $U_1, \ldots$ is a sequence of nbhds of $0 = (0, 0, \ldots)$. Let $0 \in (a_n, b_n) \subset \pi_n(U_n)$. Then $V_n = (\frac{a_n}{2}, \frac{b_n}{2})$ is a nbhd of 0 in $\mathbb{R}$, and we can let $V = \prod_n V_n$. This is a nbhd of 0 in the box topology, yet it does not contain any of the sets $U_n$. Thus the box topology is not first countable. This implies that the box topology is not 2nd countable either.

**Example** Let $X = [0, 1]^\omega$. The function $d(x, y) = \sup |x_i - y_i|$ defines a metric on $X$ as you can check. The subset $A = \{0, 1\}^\omega$ is discrete, as the metric restricts to the discrete metric on this space. Thus $A$ is first-countable, as it is a metric space. But $A$ is not second-countable. For suppose that $\mathcal{B}$ is a basis for $A$ in the discrete topology. As each $x \in A$ is isolated, $\{x\} \in \mathcal{B}$ for all $x$. But there are uncountably many points in $A$, hence uncountably many points in $\mathcal{B}$. Thus there is no countable basis for $A$.

**Theorem 23.** *A subspace of first/second-countable space is first/second-countable. A countable product of first/second-countable spaces is first/second-countable.*

*Proof.* We prove the first countability statements. The proof of the second countability statements is in Munkres.

If $A \subset X$, $x \in A$, and $X$ has countable basis $\mathcal{B} = \{U_1, \ldots\}$ at $x$. Then you can check that $\mathcal{B}_A = \{U_1 \cap A, U_2 \cap A, \ldots\}$ is a basis at $x$ for the subspace topology.

Suppose that $x \in \prod_n X_n$, where each space $X_n$ is first-countable. Let $\mathcal{B}_n$ be a basis for $X_n$ at $x_n$, the $n$th coordinate of $x$. Suppose that $J \subset \mathbb{N}$ is finite. Denote by $\mathcal{B}_J$ the collection of all $\prod_n U_n$, where $U_n = X_n$ if $n \notin J$ and $U_n \in \mathcal{B}_n$ if $n \in J$. Then $\mathcal{B}_J$ is countable, as it is in bijection with the set $\prod_{n \in J} \mathcal{B}_n$, a finite product of countable sets. The collection $\mathcal{B} = \cup \mathcal{B}_J$, where the union is taken over all finite subsets $J$ of $\mathbb{N}$, is therefore countable, as there are countably many finite subsets of $\mathbb{N}$. Claim: $\mathcal{B}$ is a basis for $\prod_n X_n$ at $x$. For take a basis element containing $x$, say $V = \prod_n V_n$, such that for the finite set $J \subset \mathbb{N}$ we have $V_n = X_n$ if $n \notin J$ and $V_n$ open in $X_n$ for all $n$. Then $x_n \in V_n$, so for each $n \in J$ we can take $U_n \in \mathcal{B}_n$ such that $x_n \in U_n \subset V_n$. Then setting $U_n = X_n$ for $n \notin J$, we obtain $x \in \prod_n U_n \in \mathcal{B}$.

□

# Separability and Lindelofocity

**Definition** A subset $A \subset X$ is dense if $\overline{A} = X$. That is, every non-empty open subset contains a point of $A$.
**Definition** A space is separable if there is a countable dense subset $Y$.
**Example** The real line, and all Euclidean spaces, are separable. For $\mathbb{Q}^n \subset \mathbb{R}^n$ is dense, as you can check using density of $\mathbb{Q}$ in $\mathbb{R}$ and the definition of the product topology.
**Definition** A space is Lindelof if every open cover of $X$ contains a countable subcover.

**Theorem 24.** *Suppose that $X$ is second-countable. Then $X$ is Lindelof and separable.*

*Proof.* Let $\{U_n\}$ be a countable base for the topology on $X$. Then if $x_n \in U_n$ is an arbitrary point for each $n$, $\{x_n\}$ is a dense subset. Suppose that $A = \{V_i\}$ is an open cover of $X$. For each $i$, let $A_i = \{U_n : U_n \subset V_i\}$. For each $n$, if it is possible, take $i_n$ such that $V_{i_n}$ contains $U_n$. If it is not possible, don't. Then $\{V_{i_n}\}$ is a countable subcollection of $\{V_i\}$ that covers $X$. For if $x \in X$, we must have $x \in U_n$ for some $n$. Then $x \in V_{i_n}$. Thus $\cup V_{i_n} = X$. □

# Countability Properties of $\mathbb{R}_\ell$

We show that $\mathbb{R}_\ell$ is first-countable, Lindelof, separable, and not second-countable.

If $x \in \mathbb{R}_\ell$ is given, the collection of nbhds $\{[x, x + 1/n)\}$ is a countable basis at $x$ as you can check. Thus $\mathbb{R}_\ell$ is first countable.

$\mathbb{R}_\ell$ is not second-countable.

Every basis element $[a, b)$ contains an open interval $(a, b)$, hence contains a rational number. Thus the collection of rational numbers $\mathbb{Q} \subset \mathbb{R}_\ell$ is a countable dense subset.

Let $\{U_i\}$ be a cover of $\mathbb{R}_\ell$ by basis elements $U_i = [a_i, b_i)$. Consider the collection of open sets $\{(a_i, b_i)\}$. Let $C = \cup(a_i, b_i)$.

Suppose that $x \in \mathbb{R} - C$. Then $[x, x + \epsilon)$ is one of the sets $[a_i, b_i)$ for some $i$, as $x$ is not contained in any of the sets $(a_i, b_i)$ and $\{[a_i, b_i)\}$ is a cover of $\mathbb{R}_\ell$. We can pick a rational number $q_x \in [x, x + \epsilon)$ for each $x \in \mathbb{R} - C$. Note that $[x, x + \epsilon)$ contains no other points of $\mathbb{R} - C$. For if $y \in [x, x + \epsilon) - \{x\}$, then $y \in (x, x + \epsilon)$. Thus if $x < y$ then $q_x < q_y$ for if $q_y \leq q_x$, then we would have $x < y < q_y \leq q_x$, so that $y \in (a_i, b_i) \subset C$. Thus $\mathbb{R} - C$ is countable.

The set $\mathbb{R} - C$ can be covered by a countable subcollection of $\{U_i\}$. The set $C = \cup(a_i, b_i)$ is a collection of real numbers. Consider $C$ as a subspace of $\mathbb{R}$ with std top. Then $C$ is second countable covered by $\{(a_i, b_i)\}$, a collection of open sets. So we can take a countable subcover $\{(a_n, b_n)\}$ of $C$, as $C$ is second countable. Now we can a take a countable subcollection of $\{U_i\}$ containing $\mathbb{R} - C$, and a countable subcollection containing $C$. The union of these two countable subcollections is countable and covers $\mathbb{R}$.

Thus $\mathbb{R}_\ell$ is Lindelof.