Math 54 - Lecture 19: Separation Axioms - Regularity and Normality

Dan Crytser

August 6, 2012

Introduction

In this lecture we introduce two separation axioms which, along with the Hausdorff axiom, are the most well-known such axioms. We provide alternate characterizations of regularity and normality in terms of nested neighborhoods. We show that subspaces and products of regular spaces are regular. We show that such behavior does *not* occur for normal spaces, as the product $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal while $\mathbb{R}_\ell$ is normal.

Regularity and Normality

Definition If $A \subset X$, a nbhd of A is an open set $U \subset X$ such that $A \subset U$.
Broadly speaking, a separation axiom describes when you can separate disjoint closed subsets by their nbhds. We have already seen two such axioms: the Hausdorff condition and the T_1 axiom, the latter of which says that one-point sets are closed. The other separation axioms we consider are all at least as strong as the T_1 axiom by definition, and you can show that they are as strong as the Hausdorff condition. There are separation axioms which are weaker than Hausdorff, but they do not yield strong theorems, so we do not consider them.
Definition A space X is called *regular* if it is T_1 and if given closed $A \subset X$ and $x \in X - A$, there are disjoint nbhds U and V of A and x, respectively. A space X is *normal* if it is T_1 and if given closed $A, B \subset X$ with $A \cap B = \emptyset$, there exist disjoint nbhds U of A and V of B.
Note: A regular space is Hausdorff, as we can take the closed set to be a point. A normal space is regular, as we can take one of the closed sets to be a point.
Example We will prove next lecture that: all metric spaces are normal and hence regular, all compact Hausdorff spaces are normal and hence regular. A homework assignment will be to show that if X is an ordered set equipped with the order topology, then X is regular.

Lemma 1. *Suppose that X is a T_1 space.*

(a) X is regular if and only if given a pt $x \in X$ and a nbhd U of x, there is a nbhd V of x satisfying $x \in V \subset \overline{V} \subset U$.

(b) *X is normal if and only if given a closed set $A \subset X$ and a nbhd U of A, there is a nbhd V of A satisfying $A \subset V \subset \overline{V} \subset U$.*

Proof. First we prove part (b). Suppose that X is normal. Let A be a closed set of X with nbhd U. Then A does not intersect the closed set $B = X - U$. Thus there are disjoint nbhds V of A and W of B. If $y \in X - U$, then W is a nbhd of y which does not intersect V, so that $y \notin \overline{V}$. Thus $\overline{V} \subset U$, and we are done. Suppose that the condition on nesting open sets holds. Let $A, B \subset X$ be disjoint closed sets not. Thus $U = X - B$ is a nbhd of A. We can take a nbhd V of A s.t. $\overline{V} \subset U$, by the condition on nesting open sets. Claim: V and $X - \overline{V}$ are disjoint nbhds of A and B respectively. For the disjointness follows $V \subset \overline{V}$, and the contrapositive yields $B = X - U \subset X - \overline{V}$. Munkres runs the proof for regular spaces: it is the same except one of the closed sets is replaced by a point. □

Subspaces and Products

Regular and Hausdorff spaces are very nicely behaved with respect to the product and subspace operations. Normal spaces are not.

Theorem 1. *A subspace of a Hausdorff/regular space is Hausdorff/regular. A product of Hausdorff/regular spaces is Hausdorff/regular.*

Proof. The Hausdorff stuff is done back in HW 3. The regularity is very similar. Suppose that Y is a subspace of regular space X. Then if $y \in Y$, $\{y\} = Y \cap \{y\}$, a closed set in the subspace topology as one-point subsets are closed in X. Thus one-point sets are closed in the subspace topology. Suppose that $y \in Y$ and $A \subset Y$ is closed in the subspace topology. Then $A = C \cap Y$ for some closed set $C \subset X$, which gives $y \notin C$. As X is regular, we can take disjoint nbhds of y, C respectively, say U and V open in X. Then $U \cap Y$ and $V \cap Y$ are disjoint nhbds of y and A in Y.

Suppose that $\{X_i\}$ is a family of regular spaces. Then each space X_i is Hausdorff, so that the product $\prod X_i$ is Hausdorff. As Hausdorff spaces are T_1, $\prod X_i$ is T_1. Suppose that $x \in U \subset \prod X_i$, where U is open in the product. We can take a basis element $\prod U_i \subset U$ containing x, so that for each i we have $x_i \in U_i$, and each U_i is open. As X_i is regular, we can for each i take an open set V_i such that $x_i \in V_i \subset \overline{V_i} \subset U_i$, where if $U_i = X_i$ we take $V_i = X_i$ as well. Then $V = \prod V_i$ is a basis element in the product topology, and we have $\overline{V} = \prod \overline{V_i} \subset \prod U_i \subset U$. Thus $\prod X_i$ is regular.

□

Two Examples: $\mathbb{R}_K$ and $\mathbb{R}_\ell^2$.

Example The space $\mathbb{R}_K$ is Hausdorff but not regular. For $\mathbb{R}_K$ contains the Euclidean topology on $\mathbb{R}$, which we know to be Hausdorff. Thus it must be Hausdorff as well. Note that as $\mathbb{R} - K$ is open in $\mathbb{R}_K$, K is closed. Claim: there do not exist nbhds U of 0 and V of $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}_K$. Suppose that $(-\epsilon, \epsilon) - K$ is a basis elt containing 0 and U is an open subset of $\mathbb{R}_K$ containing K. Take n such that $\frac{1}{n} < \epsilon$. Then U must contain an interval $(\frac{1}{n} - \delta, \frac{1}{n} + \delta)$ as U is open and $K \subset U$. As the interval $(\frac{1}{n} - \delta, \frac{1}{n})$ contains uncountably many points, we can take a point z of $(\frac{1}{n} - \delta, \frac{1}{n}) - K$. Then $z \in (-\epsilon, \epsilon) \cap U$, so that the nbhds cannot be taken disjoint.

Thus a Hausdorff space need not be regular.

Example The space $\mathbb{R}_\ell$ is normal. Let A and B be disjoint closed subsets of $\mathbb{R}_\ell$. For each $a \in A$, choose $[a, x_a)$ which does not intersect B (as $a \notin \overline{B} = B$). Do the same for B, choosing $[b, x_b)$ which does not intersect A. Then $U = \cup_a [a, x_a)$ and $V = \cup_b [b, x_b)$ are disjoint nbhds of A and B respectively. For suppose that $z \in [a, x_a) \cap [b, x_b)$. Then $b \leq z < x_a$ and $a \leq q < x_b$. If $a < b$, then $b \in [a, x_a)$. So we must have $b < a$. Then $b < a < x_b$, so that $a \in [b, x_b)$. So the nbhds are disjoint.

Example The product $\mathbb{R}_\ell^2$ is regular but not normal. The first part is simple: $\mathbb{R}_\ell$ is normal, hence regular. Thus the product of regular spaces $\mathbb{R}_\ell^2$ is regular. Proving that $\mathbb{R}_\ell^2$ is not normal is a bit trickier.

Let $L = \{x \times -x : x \in \mathbb{R}\} \subset \mathbb{R}_\ell^2$, the line through the origin with slope -1. This is a closed set in $\mathbb{R}_\ell^2$ as it is closed in the Euclidean topology on $\mathbb{R}^2$ and the product topology on $\mathbb{R}_\ell^2$ is finer than the Euclidean topology. Each subset of L is closed in L, as you showed on HW 3. As L is closed, this means that each subset of L is closed in $\mathbb{R}_\ell^2$ (closed intersect closed is closed). So for each proper non-empty subset $A \subset L$, we can find disjoint nbhds U_A of A and V_A of $L - A$ respectively (these are open sets in $\mathbb{R}_\ell^2$).

Let $D = \mathbb{Q}^2$. Then D is dense in $\mathbb{R}_\ell^2$ as you can confirm by inspecting basis elements. Thus each open set in $\mathbb{R}_\ell^2$ must have non-empty intersection with D. Define a map:

$$\theta(A) = \begin{cases} D \cap U_A & \emptyset \subsetneq A \subsetneq L \\ \emptyset & A = \emptyset \\ D & A = L \end{cases}$$

This is a map $\theta : \mathcal{P}(L) \to \mathcal{P}(D)$. We show that it is an injection. As $\mathcal{P}(L)$ is in bijection with $\mathcal{P}(\mathbb{R})$ and $\mathcal{P}(D)$ is in bijection with $\mathcal{P}(N)$ which is in bijection with $\mathbb{R}$, this is a contradiction to a fact of set theory: there is no injection $\rho : \mathcal{P}(X) \to X$ for any non-empty set X.

Suppose that $A \subset L$ be proper. As $D \cap U_A$ does not intersect V_A, it does not contain those elements of D which belong to the nbhd V_A. Thus $\theta(A) \neq D$, and $\theta(A)$ cannot equal $\emptyset$.

So all we have to consider is whether $\theta(A)$ can equal $\theta(B)$ for distinct proper non-empty subsets A and B of L. Suppose that A and B are distinct, so that there exists WLOG $x \in A - B$. Then $x \in U_A$ and $x \in X - B \subset V_B$. Thus $x \in U_A \cap V_B$, and so this open set must be non-empty and therefore contain a point of D. But then this point of D, say d, is in $\theta(A) = U_A \cap D$ but not in $\theta(B) \subset X - V_B$. So $\theta(A) \neq \theta(B)$.