# Math 54 - Lecture 20: Normal Spaces and Urysohn's Lemma

Dan Crytser

August 10, 2012

## Introduction

In this section we expand our collection of normal spaces to include all second countable regular spaces and compact Hausdorff spaces. We show that locally compact Hausdorff spaces are regular. Then we prove the Urysohn lemma, which, given disjoint closed subsets $A$ and $B$ of a normal space $X$, enables us to construct a function $f : X \to [0, 1]$ which is equal to 0 on $A$ and equal to 1 on $B$.

## Second Countable Regular Spaces

**Theorem 25.** *Let $X$ be a second countable regular space. Then $X$ is normal.*

*Proof.* Let $X$ be a regular space with countable basis $\mathcal{B}$. Let $A$ and $B$ be disjoint closed subsets. We construct disjoint nbhds of $A$ and $B$ respectively. For each $x \in A \subset X - B$, we can take a nbhd $U$ of $x$ such that $\overline{U} \subset X - B$, by regularity. Then choose an element $B_x$ of $\mathcal{B}$ such that $x \in B_x \subset U$. The collection of all such $B_x$ covers $A$, and as $\mathcal{B}$ is countable, we can enumerate it as $\{U_n\}$, where each $U_n$ is an open set whose closure $\overline{U_n} \subset X - B$. Choose a countable collection $\{V_n\}$ of open sets which covers $B$ such that $\overline{V_n} \subset X - A$ for all $n$.

Let $U_n' = U_n - \cup_{i=1}^n \overline{V_i}$. Let $V_n' = V_n - \cup_{i=1}^n \overline{U_i}$. Note that if $n \leq m$, then $U_n'$ and $V_m'$ are disjoint, as $U_n' \subset U_n$ and $V_m' \subset X - U_n$. Similarly if $n \geq m$, then $U_n'$ and $V_m'$ are disjoint. Thus the sets $U_n'$ and $V_m'$ are disjoint for all $m, n$.

Note that the collection $\{U_n'\}$ is an open cover for $A$, for if $x \in A$, $x$ belongs to some $U_n$, and as each of the sets $\overline{V_i}$ do not meet $A$, $x \in U_n'$. Similarly $\{V_n\}'$ is an open cover for $B$. Then $U = \cup U_n'$ and $V = \cup V_n'$ are disjoint by the distributive law, and they are nbhds of $A$ and $B$ respectively. □

# Compact Hausdorff Spaces

**Theorem 26.** *Let $X$ be a compact Hausdorff space. Then $X$ is normal.*

*Proof.* We have already basically finished this in assignment 7. For if $X$ is compact, and $A$ and $B$ are closed, then $A$ and $B$ are disjoint compact subsets of $X$. As $X$ is Hausdorff, we can by problem 26.5 choose disjoint nbhds $U$ and $V$ of $A$ and $B$ respectively. □

**Corollary 1.** *Let $X$ be a locally compact Hausdorff space. Then $X$ is regular.*

*Proof.* This is an example of the utility of the one-point compactification. Let $Y$ be the compact Hausdorff space from theorem 29.1. Then $X$ is embedded as a subspace of $Y$. Note that as $Y$ is compact Hausdorff, $Y$ is normal by the preceding theorem. Thus $Y$ is also regular. As $X$ is a subspace of the regular space $Y$, $X$ is regular also. □

**Corollary 2.** *Let $X$ be a second-countable locally compact Hausdorff space. Then $X$ is normal.*

*Proof.* Suppose $X$ is such. Then $X$ is regular and second countable, by the previous two theorems. Thus $X$ is normal by the theorem on regular secound countable spaces. □

**Example** There is an example which shows that a locally compact Hausdorff space doesn't have to be normal. I erroneously asserted that $\prod_J (0,1)$ is locally compact Hausdorff, but not normal if $J$ is uncountable. However, this space is not locally compact. The primary counterexample is called the Deleted Tychonoff Plank, which involves ordinal numbers (a special type of order topology).

# The Urysohn Lemma

Now we arrive at a result which shows how powerful the normality condition is. It's called a lemma for historical reasons – in fact it is a powerful and interesting result in its own right.

**Theorem 27.** *(Urysohn's Lemma) Let $X$ be a normal space, let $A$ and $B$ be disjoint subsets of $X$. Then there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

*Proof.* Let $A$ and $B$ be disjoint closed subsets of the normal space $X$. We construct the desired function. The proof goes in stages

1. Let $P = [0,1] \cap \mathbb{Q}$ be the collection of all rational numbers in $[0,1]$. In the first stage, we construct a family of open sets $\{U_p\}_{p \in P}$ such that if $p, q \in P$ and $p < q$, then $\overline{U_p} \subset U_q$, and such that $A \subset U_p$ for all $p \in P$.

   You can enumerate the rationals as $1, 0, \ldots$, so that the first two numbers in the sequence are 1 and 0. The $n$th term of the sequence is denoted by $p_n$. NOTE: we index the open sets by the rational numbers themselves, not by the integers which keep track of the rational numbers in this sequence.

   Let $U_1 = X - B$. Note that this is a nbhd of $A$, and as $A$ is closed and $X$ is normal, we can take a nbhd $U_0$ such that $A \subset U_0 \subset \overline{U_0} \subset U_1$.
   Let $P_n = \{1, 0, \ldots, p_n\}$. Suppose that $U_p$ is defined for all $p \in P_n$ such that if $q, r \in P_n$ with $q < r$, then $\overline{U_q} \subset U_r$. Assume that $n \geq 2$.

   Let $r = p_{n+1}$, the next rational number in the sequence. We know that $r$ is not 1 or 0, so we must be able to find in the finite ordered set $P_n \cup \{r\}$ an immediate predecessor and an immediate successor, say $p$ and $s$ in $P_n$. Then $p < r < s$, so that $\overline{U_p} \subset U_s$. By normality, we can find an open set $U_r$ containing $U_p$ satisfying $\overline{U_r} \subset U_s$.
   Check: if we define $U_r$ this way, and if $q, q' \in P_{n+1}$ such that $q < q'$, then $\overline{U_q} \subset U_{q'}$. As we can do this for all $n$, we have defined $U_p$ for all $p \in P$.

2. The second step is simple: we extend the definition of $U_p$ to all rational numbers by setting $U_p = \emptyset$ for $p$ negative rational and $U_p = X$ for all $p > 1$.
   Check: if $p, q \in \mathbb{Q}$, and $p < q$, then $\overline{U_p} \subset U_q$. All the containments you have to check are simple: either the empty set is contained in everything, or the total set contains everything.

3. Now we define the function $f$. For $x \in X$, let $\mathbb{Q}(x) = \{p \in \mathbb{Q} : x \in U_p\}$. Note that $\mathbb{Q}(x)$ is a non-empty collection of non-negative numbers, as $x \in U_2$ for all $x$ and $U_p = \emptyset$ for $p$ negative. Thus, as $\mathbb{Q}(x)$ is a non-empty set of real numbers bounded from below, we can define $f(x) = \inf \mathbb{Q}(x)$, a well-defined function $f : X \to \mathbb{R}$.

4. Now we show that $f$ vanishes on $A$, attains the value 1 on $B$, and is continuous. Suppose that $x \in A$. As $A \subset U_0$, we have $0 \in \mathbb{Q}(x)$. Then $0 = \inf \mathbb{Q}(x)$, as $\mathbb{Q}(x)$ is a collection of non-negative numbers. Thus $f(x) = 0$. Suppose that $x \in B$. If $p \leq 1$, then $U_p \subset U_1 = X - B$, so that $x \notin U_p$. Thus $\mathbb{Q}(x) = (1, \infty)$. So $f(x) = 1$.
The tricky part is in showing that $f$ is continuous. First we single out two facts that make our life easier:

   (a) If $x \in \overline{U_r}$, then $f(x) \leq r$. For if $r < p$, then $\overline{U_r} \subset U_p$, so that $x \in U_p$. Thus $(r, \infty) \subset \mathbb{Q}(x)$, and so $f(x) \leq r$, as $f(x)$ is a lower bound for $\mathbb{Q}(x)$.

   (b) If $x \notin U_r$, then $f(x) \geq r$. Note that if $s < r$, then $U_s \subset U_r$. Thus if $f(x) \notin U_r$, then $\mathbb{Q}(x) \subset (r, \infty)$. Thus $r$ is a lower bound for $\mathbb{Q}(x)$, and we must have $f(x) \geq r$.

   Now suppose that $x \in X$ and $(c, d)$ is a nbhd of $f(x)$ in $\mathbb{R}$. We construct a nbhd $U$ of $x$ such that $f(U) \subset (c, d)$. Then $c < f(x) < d$. Pick rational numbers $p \in (c, f(x))$ and $q \in (f(x), d)$.

   Claim: $U = U_q - \overline{U_p}$ is a nbhd of $x$ satisfying $f(\overline{U}) \subset [p, q] \subset (c, d)$. For we can take the contrapositive of 1 and 2 above to see that $x \notin \overline{U_p}$ and $x \in U_q$. This is the intersection $U_q \cap (X - \overline{U_p})$, an open set. Suppose that $x \in U$. Then $x \in U_q \subset \overline{U_q}$, so that $f(x) \leq q$. Also $x \notin \overline{U_p} \supset U_p$, so that $f(x) \geq p$. Thus $f(x) \in [p, q]$ and we are done.

□

**Corollary 3.** *Let $X$ be a $T_1$ space. Then $x$ is normal if and only if given two disjoint closed subsets $A$ and $B$ of $X$, there exists a continuous function $f : X \to [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

*Proof.* The "only if" direction is Urysohn's lemma. Suppose that the functions described exist, and let $A$ and $B$ be disjoint closed subsets of $X$. Take $f$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Then $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$ are disjoint nbhds of $A$ and $B$. As $X$ is assumed to be $T_1$, $X$ is normal. □