# MATH 54 - LECTURE 3

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## Introduction

In this lecture we introduce the idea of a topological space and explore numerous examples. We also examine bases for spaces, which are the topological analogue to the open balls we looked at in $\mathbb{R}^n$.

## Euclidean Space

Remember that we defined a subset $U \subset \mathbb{R}^n$ to be open if about any pt $x \in U$ we could inscribe some open ball $B(x, \epsilon) \subset U$. We know that $\mathbb{R}^n$ and $\emptyset$ are both open, the first because $\mathbb{R}^n$ contains all the open balls and the second because there are no points $x$ to worry about ($\emptyset$ is *vacuously open* in fancy language). We proved the following two facts:

(1) The union $\cup_{i \in I} U_i$ of any (arbitrary) collection $\{U_i\}_{i \in I}$ of open sets is itself an open set.
(2) The intersection $U \cap V$ of any pair of open sets $U, V$ is again open. (In fact, the intersection of any finite collection of open sets is again open).

## Topological Spaces

We mimic these properties in the definition of a general topological space.
**Definition** Let $X$ be a set, and let $\tau$ be a collection of subsets of $X$ which satisfies the following conditions:

(1) $X, \emptyset \in \tau$
(2) If $\{U_i\}_{i \in I}$ is a collection of subsets of $X$, each of which is in $\tau$, then $\cup_{i \in I} U_i \in \tau$. That is
$$\{U_i\}_{i \in I} \subset \tau \to \cup_{i \in I} U_i \in \tau.$$
(3) If $U_1, U_2, \ldots, U_n \in \tau$, then $\cap_{i=1}^n U_i \in \tau$, i.e. any finite intersection of sets in $\tau$ is also in $\tau$.

Then we call $\tau$ a *topology* on $X$, and we call the pair $(X, \tau)$ a topological space. The sets in $\tau$ are called the *open sets* of the topological space. When the topology on $X$ is understood we will often abuse notation and simply call $X$ a topological space. Sometimes we refer to the topological space $(X, \tau)$ as $X$ equipped with the topology $\tau$.

We call the rules above the "axioms for a topology." In order to show that the third axiom is true, you only need to show that if $U, V \in \tau$, then $U \cap V \in \tau$, for you can write $U_1 \cap U_2 \cap \ldots \cap U_n = (U_1 \cap U_2 \cap \ldots \cap U_{n-1}) \cap U_n$ for $n > 2$, and keep breaking the set in parentheses into an intersection of a shorter list of sets. This is a big definition. When a big definition comes along you must look for a large collection of interesting examples.

**Example** Let $X = \mathbb{R}^n$ and let $\tau$ be the collection of all open subsets of $\mathbb{R}^n$ as determined by inclusion of open balls. Then the two propositions stated above show that $(\mathbb{R}^n, \tau)$ is a topological space. If we refer to $\mathbb{R}^n$ as a topological space in the future, it is assumed that this topology is the one we are using, unless otherwise stated. This is the *standard* or *Euclidean* topology on $\mathbb{R}^n$.

**Example** Let $X$ be any set. Then the collection $\{X, \emptyset\}$ is a topology on $X$, as you can check. This is a really dumb topology (IMO) called the *indiscrete* topology on $X$.

**Example** Let $X$ be any set. Then the collection $\mathcal{P}(X)$ of all subsets of $X$ is a topology on $X$, because the union and intersection of any collection of subsets of $X$ is again a subset of $X$. This topology arises more frequently and is known as the *discrete* topology. Note that the discrete topology is the only topology in which all of the one-point subsets $\{x\} \subset X$ for $x \in X$ are open.

Let $X$ be any set. Then let $\tau = \{U \subset X : X \setminus U \text{ is finite or all of } X\}$. Let's go through the rules for a topology one by one:

(1) Is $X \in \tau$? The complement $U = X \setminus X = \emptyset$ is finite, so the complement of $X$ is finite (in fact, empty) and so $X \in \tau$.
(2) Is $\emptyset \in \tau$? The complement $U = X \setminus \emptyset = X$ is all of $X$, so that $\emptyset \in \tau$.
(3) Suppose that $\{U_i\}_{i \in I}$ is a collection of sets in $\tau$, so that $X \setminus U_i$ is finite for every $i \in I$. We want to confirm that $\cup_{i \in I} U_i$ is also a set in $\tau$. We can write

$$X \setminus (\cup_{i \in I} U_i) = \cap_{i \in I} X \setminus U_i.$$

If $I = \emptyset$ (a stupid degenerate case) then the intersection on the right is all of $X$. If $I \neq \emptyset$, then the right-hand intersection is an intersection of finite sets, hence a finite set itself. Thus the complement of $\cup_{i \in I} U_i$ is finite, and $\cup_{i \in I} U_i$ is in $\tau$.
(4) Suppose that $U$ and $V$ are both in $\tau$. Then

$$X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V).$$

If either of $X \setminus U$ or $X \setminus V$ is all of $X$, then the right-hand union above is all of $X$ as well. So assume that both $X \setminus U$ and $X \setminus V$ are finite. Then their union is again finite, as the union of finite sets is finite. Thus $X \setminus (U \cap V)$ is either all of $X$ or finite. So $U \cap V$ is in $\tau$.

We call this the *cofinite* (short for "complement is finite") topology on $X$. Note that if $X$ is finite, then all subsets $U \subset X$ have finite complement, and the cofinite topology is just the discrete topology.
**Question**: If we give $\mathbb{N}$ the cofinite topology, is the set $E = \{2, 4, 6, \ldots\}$ of all even positive integers an open subset?
**Example** Now let $X$ be any *uncountable* set. We define the *co-countable* topology on $X$ as follows: a subset $U \subset X$ is open in the co-countable topology if $U = \emptyset$ or if the complement $X \setminus U$ is countable. You can check that this defines a topology on $X$, using deMorgan's laws and the fact that the intersection $\cap_{i \in I} U_i$ of a collection $\{U_i\}_{i \in I}$ is countable (as it is a subset of each of the countable sets $U_i$), and the fact that for $U, V$ countable sets, the union $U \cup V$ is also countable, which is a special case of the theorem on countable unions of countable intersections we proved in the second lecture.

## Finer and Coarser

**Definition** If $(X,\tau)$ and $(X,\tau')$ are topological spaces (with the same underlying set, we say that the topology $\tau'$ is *finer* than $\tau$ if

$$\tau \subset \tau'.$$

That is, $\tau'$ is finer than $\tau$ if every open set in $(X,\tau)$ is also an open set in $(X,\tau')$. If $\tau'$ is finer than $\tau$ we say that $\tau$ is coarser than $\tau'$. If neither $\tau$ nor $\tau'$ is finer than the other, than we say that the topologies are incomparable.

**Example** If $X$ is a set and $\tau$ is a topology on $X$, and $\tau_d$ denotes the discrete topology (i.e. $\tau_d = \mathcal{P}(X)$) then $\tau \subset \tau_d$. That is, the discrete topology is the finest topology available on any set $X$

**Example** On the other hand, it is easy to see by the first axiom for a topology that the indiscrete topology is coarser than any other topology on $X$. That is, the indiscrete topology is the coarsest topology available on any set $X$.

**Example** Now let's do an interesting example of finer and coarser topologies. Let the real number number line be equipped with the standard topology. Call this $(\mathbb{R},\tau_s)$. We can consider another topology on $\mathbb{R}$, called $\tau_\ell$ which is given by the following rule: $U \subset \mathbb{R}$ is open if $U$ is empty or if for every $x$ in $U$ you can draw a half open interval $[a,b)$ so that $x \in [a,b) \subset U$. (We call this the *lower limit topology*, and when we wish to refer to $\mathbb{R}$ equipped with this topology we will abbreviate it as $\mathbb{R}_l$). We allow $x = a$ in the definition, so that for example the set $[0,1)$ is open in $\tau_l$, whereas $[0,1)$ is not open in $\tau_s$, because I cannot inscribe an open ball around 0 (that is, an interval $(-\epsilon,\epsilon)$) which contains no negative numbers. Thus the two topologies on Claim: $\tau_s \subset \tau_l$. We need to show that if $U \subset \mathbb{R}$ is open in $\tau_s$, then $U$ is open in $\tau_l$. So suppose that $U \in \tau_s$. If $x \in U$, then as $U$ is open in $\tau_s$ we can produce an open ball $(x-\epsilon, x+\epsilon) \subset U$. Now $[x, x+\epsilon) \subset U$. Thus about an arbitrary point $x \in U$ we have produced a half-open interval contained in $U$. So $U$ is open in $\tau_l$.

## Basis

Recall: a set $U \subset \mathbb{R}^n$ is open if and only if about each point $x \in U$ we can inscribe an open ball $B(x,\epsilon)$ within $U$. We can mimic this behavior in arbitrary topological spaces:

**Definition** Let $X$ be a set and let $\mathcal{B} = \{B_i\}_{i\in I}$ be a collection of subsets of $X$. We say that $\{B_i\}_{i\in I}$ is a basis if the following conditions hold:

(1) For each $x \in X$, there exists $B_i$ such that $x \in B_i$. In other words, $X = \cup_{i\in I} B_i$.
(2) If $x \in B_i \cap B_j$ for some sets $B_i, B_j \in \mathcal{B}$, then there exists $B_k \in \mathcal{B}$ such that $x \in B_k \subset B_i \cap B_j$.

**Definition** Given a set $X$ and a basis $\mathcal{B}$, the *topology generated by the basis $\mathcal{B}$* is defined as follows: a set $U \subset X$ is open in the topology gen'd by $\mathcal{B}$ if for each $x \in U$, there is a $B \in \mathcal{B}$ such that $x \in B \subset U$.

**Theorem 1.** *Let $(X,\tau)$ be a topological space and let $\mathcal{B}$ be a basis for the topology $\tau$. Then every open subset of $(X,\tau)$ is a union of sets in $\mathcal{B}$.*

*Proof.* You replicate the proof that every open set in Euclidean $n$-space is a union of open balls precisely. ☐

**Example** If $X$ is an ordered set under $<$, then the collection $\{(a,b) : a < b\} \cup \{X\}$ is a basis for the order topology $\tau_<$. **Example** If $X$ is any set, the collection $\{\{x\} : x \in X\}$ is a basis for the discrete topology on $X$.

**Example** The set of open intervals $\{(a,b) : a < b\}$ in $\mathbb{R}$ is a basis, as previously indicated. It generates the standard topology on $\mathbb{R}$.

**Example** The set of half-open intervals $\{[a,b) : a < b\}$ in $\mathbb{R}$ is a basis. It generates the topological space $(\mathbb{R}, \tau_l) = \mathbb{R}_l$, the lower limit topology.

**Example** The set of closed intervals $\{[a,b] : a < b\}$ is NOT a basis for any topology. For example, the sets $[0,1]$ and $[1,2]$ both contain 1, but there is no closed interval $[c,d]$ containing 1 with $[c,d] \subset [0,1]$ and $[c,d] \subset [1,2]$. If we instead use $\{[a,b] : a \leq b\}$, then the collection is a basis, and the topology it generates on $\mathbb{R}$ is the discrete topology.

**Example** Let $X = \mathbb{R}$ and let $K = \{\frac{1}{n} : n \in \mathbb{N}\}$. Let $\mathcal{B}_K = \{U : U = (a,b) \text{ or } U = (a,b) \setminus K\}$, the collection of all open intervals in $\mathbb{R}$ along with the collection of all open intervals with the set $K$ deleted (note that many of the intervals $(a,b)$ in $\mathbb{R}$ don't intersect $K$ at all, so that $(a,b) \setminus K = (a,b)$.) Claim: $\mathcal{B}_K$ is a basis. Proof of claim: As $\mathcal{B}_K$ contains all the open intervals $(a,b)$, it has to cover all of $\mathbb{R}$ (for example $x \in (x-1, x+1)$ for any $x \in \mathbb{R}$). Let $B_1$ and $B_2$ be elements of $\mathcal{B}_K$, and suppose that $x \in B_1 \cap B_2$. We need to produce a $B_3$ such that $x \in B_3 \subset B_1 \cap B_2$. If neither of the sets $B_1$ and $B_2$ has the form $(a,b) \setminus K$, they are both open intervals, say $B_1 = (a,b)$ and $B_2 = (c,d)$, and we can take $p = \max\{a,c\}$ and $q = \min\{b,d\}$ and obtain $x \in (p,q) \subset B_1 \cap B_2$. If $B_1 = (a,b) \setminus K$, we have $x \in ((a,b) \cap (c,d)) \setminus K$ for some $(a,b)$ and $(c,d)$. Then taking $p, q$ as before we obtain $x \in (p,q) \setminus K \subset B_1 \cap B_2$. Thus the collection $\mathcal{B}_K$ is a basis.

When $\mathbb{R}$ is equipped with this topology, we denote it by $\mathbb{R}_K$. Claim: the topology generated by $\mathcal{B}_K$ is strictly finer than the standard topology on $\mathbb{R}$ (generated by $\{(a,b)\}$). Proof of claim: suppose that $U \subset \mathbb{R}$ is open in the standard topology. We show that $U$ is also open in $\mathbb{R}_K$. Suppose that $x \in U$. We can find $(a,b)$ such that $x \in (a,b) \subset U$ by the definition of the standard topology on $\mathbb{R}$. But $(a,b) \in \mathcal{B}_K$ by definition, so that $U$ is open in $\mathbb{R}_K$. Now the set $(-1,1) \setminus K$ is open in $\mathbb{R}_K$, because it is a basis element. However, we can see that $(-1,1) \setminus K$ is not open in $\mathbb{R}$ equipped with the standard topology. For $0 \in (-1,1) \setminus K$, and yet any open interval $(-\epsilon, \epsilon)$ about 0 must intersect the set $K = \{\frac{1}{n} : n \in \mathbb{N}\}$, for example by taking $n$ such that $\frac{1}{n} < \epsilon$. So the topology generated by $\mathcal{B}_K$ is strictly finer than the standard topology on $\mathbb{R}$.