MATH 54 - LECTURE 4

DAN CRYTSER

Introduction

In this lecture we review properties and examples of bases and subbases. Then we consider ordered sets and the natural "order topology" that one can lay on an ordered set, which makes all the open intervals (a, b) into open sets.

Bases, ctd.

Theorem 1. *Let X be a topological space, and let $\mathcal{B}$ be a basis for the topology on X. Then a subset $U \subset X$ is open if and only if $U = \cup_{i \in I} B_i$, for some collection $\{B_i\}_{i \in I} \subset \mathcal{B}$. That is, the open sets are precisely the unions of basis elements.*

Proof. Suppose that U is open. For each $x \in U$, select B_x such that $x \in B_x \subset U$, by the definition of a basis. Then $U = \cup_{x \in U} B_x$ is a union of basis elements. The other direction follows from two facts: each basis element is open, and a union of open sets is again open. □

Example Let $\mathcal{B} = \{(a, b) : a < b\}$ be the collection of all open intervals in $\mathbb{R}$. Then $\mathcal{B}$ is a basis and it generates the standard topology on $\mathbb{R}$.

Example Let $\mathcal{C} = \{[a, b) : a < b\}$ be the set of all half-open intervals in $\mathbb{R}$. Claim: $\mathcal{C}$ is a basis. For $a \in [a, a + 1)$, so that the proposed basis does contain $\mathbb{R}$ in its union. Suppose that $x \in [a, b) \cap [c, d)$. Then $a, c \leq x < b, d$. Taking $p = \max\{a, c\}$ and $q = \min\{b, d\}$ we see from the inequality that $p < q$, and $x \in [p, q) \subset [a, b) \cap [c, d)$. Thus if a point lies in the intersection of two $\mathcal{C}$ elements, we know how to inscribe a third element which contains the point and lies entirely within the intersection. Thus $\mathcal{C}$ is a basis. We call the topological space generated by this basis $\mathbb{R}_\ell$, the lower limit topology. The topology that $\mathcal{C}$ generates contains the standard topology on $\mathbb{R}$. To see this, we only need to show that the basis elements of the standard topology are open in $\mathbb{R}_\ell$. So let (a, b) be a basis element, and let $x \in (a, b)$. We need to find a basis element $[p, q)$ in $\mathcal{C}$ which contains x and which lies within (a, b). If we take $p = x$ and $q = b$, we see that $[x, b) \subset (a, b)$ works. So every basis element in $\mathbb{R}$ in the standard topology is open in $\mathbb{R}_\ell$, and applying the union of basis elements theorem above we see that all the open sets in $\mathbb{R}$ in the standard topology are open. Now we can see that the topology of $\mathbb{R}_\ell$ strictly contains the standard topology by exhibiting a set which is open in $\mathbb{R}_\ell$ which is not open in the standard topology.

Claim: $[0, 1)$ is not open in the standard topology on $\mathbb{R}$. For it is impossible to draw an open interval around 0 which does not contain any negative numbers, and hence no open interval around 0 is contained in $[0, 1)$.

Example Let $K = \{1/n : n \in \mathbb{N}\}$. Let $\mathcal{B}''$ be the collection of sets of the form (a, b) or $(a, b) \setminus K$, for $a < b$. That is $\mathcal{B}$" contains all open intervals and all open intervals with the fractions in K deleted. Claim: $\mathcal{B}$" is a basis. To see that $\mathcal{B}$" contains $\mathbb{R}$ as its union, just notice that the open intervals cover $\mathbb{R}$ as their union. Suppose that $B_1, B_2 \in \mathcal{B}$" and

$x \in B_1 \cap B_2$. If both B_1 and B_2 are open intervals without the points deleted, the same argument as before, where you take the max of the left endpoints and the minimum of the right endpoints, produces another open interval which lies entirely within the intersection. If either of B_1 or B_2 has K deleted, then produce the small interval as in the previous sentence, then delete K from it. Thus $\mathcal{B}$" is a basis. We call the topological space generated by $\mathcal{B}$" by $\mathbb{R}_K$, the K-topology on $\mathbb{R}$.

Claim: the topology of $\mathbb{R}_K$ contains the standard topology on $\mathbb{R}$. For each open interval (a, b) is open in $\mathbb{R}_K$, and so we are done by the theorem on basis.

Claim: the topology of $\mathbb{R}_K$ is strictly larger than the standard topology. For the set $(-1, 1) \setminus K$ is open in $\mathbb{R}_K$, being a basis element. However, $(-1, 1) \setminus K$ is not open in $\mathbb{R}$ with the standard topology, for every open interval around 0 must contain some element of the set $\{1/n : n \in \mathbb{N}\}$.

Sub-bases

Definition Let X be a set. Then a sub-basis is a collection $\mathcal{S}$ such that $\cup_{S \in \mathcal{S}} S = X$. The topology generated by $\mathcal{S}$ is defined by the rule: $U \subset X$ is open if for each $x \in U$ we can find $S_1, S_2, \ldots, S_n$ such that $x \in S_1 \cap S_2 \ldots S_n \subset U$.

Definition Let X be a topological space with topology τ. Then $\mathcal{S} \subset \tau$ is a sub-basis for X if for each open set U and each $x \in U$, there exist finitely many $S_1, \ldots, S_n \in \mathcal{S}$ such that $x \in S_1 \cap S_2 \ldots \cap S_n \subset U$.

Example A stupid example of a sub-basis is any basis.

Example Let $\mathcal{S} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$, the set of all infinite rays in $\mathbb{R}$. Then $\mathcal{S}$ is clearly not a basis for the standard topology on $\mathbb{R}$. For example: $(0, 1)$ is open in $\mathbb{R}$, but I can't find an infinite ray containing $1/2$ which is contained in $(0, 1)$. But $\mathcal{S}$ is a sub-basis, for $(a, b) = (a, \infty) \cap (-\infty, b)$, so if U is an open set containing the point x, and $x \in (a, b) \subset U$, we have $x \in (-\infty, b) \cap (a, \infty) \subset U$.

Ordered Sets

Let X be a set and $<$ a total ordering on X. That is, $<$ is a relation with the properties:

(1) If $a < b$ and $b < c$ then $a < c$
(2) For no a does $a < a$
(3) If $a < b$ then it is not the case that $b < a$.

An example of an ordered set is $(\mathbb{N}, <)$ with the standard ordering, or $(\mathbb{R}, <)$ with the same standard ordering. Refer to the notes for more information on ordered sets.

Definition If X is an ordered set under $<$, then we say that $a_0 \in X$ is the smallest element of X if $a_0 \leq b$ for all $b \in X$. Similarly, $b_0 \in X$ is the largest element of X if $a \leq b_0$ for all $a \in X$.

The Order Topology

We define a basis on X by the following

Definition Let $\mathcal{B}$ be the collection of all subsets of X of the following types:

(1) All open intervals $(a, b) = \{x : a < x < b\}$.
(2) All intervals $[a_0, b)$ where a_0 is the smallest element of X, if such exists.
(3) All intervals of the form $(a, b_0]$, where b_0 is the largest element of X, if such exists.

One of the homework problems will be to show that the previous definition is actually a basis.

Example Let $X = \mathbb{R}$ be given the standard ordering. We show that the order topology coincides with the standard Euclidean topology. It suffices to show that a basis set for the order topology is open in the standard topology, and that a basis set in the standard topology is open in the order topology. Let U be a basis set for $\mathbb{R}$ in the order topology. As $\mathbb{R}$ does not have any largest or smallest element in the standard ordering, $U = (a, b)$ for some $a < b$. But this set is open in the standard topology. Let U be a basis set for $\mathbb{R}$ in the standard topology. Then $U = (x - \epsilon, x + \epsilon)$ for some $x \in \mathbb{R}$, $\epsilon > 0$. But this is open in the order topology.

Example Let $X = \mathbb{N}$ with the standard ordering. We show that the order topology is discrete. All we have to do is show that the singleton sets $\{n\}$ for $n \in \mathbb{N}$ are open in the order topology. Note that $\{n\} = (n - 1, n + 1)$ for any integer $n > 1$. Thus $\{n\}$ is open for $n > 1$. The set $\{1,\}$ equals $[1, 2)$, and 1 is the smallest element, so by the definition of the order topology $\{1\}$ is a basis element and is therefore open.

Example Let $X = \mathbb{R} \times \mathbb{R}$ with the dictionary ordering. That is, $(x, y) < (x', y')$ if $x < x'$ or if $x = x'$ and $y < y'$. (For points P,Q, we have $P < Q$ if Q is to the right of P or directly above P). We consider the order topology on X with this ordering. There is no largest or smallest element in this ordering as (note that $(x - 1, y) < (x, y)$ for all $(x, y) \in X$). Thus we only have to consider the open intervals (P, Q) for distinct points P and Q in the plane (careful not to confuse open intervals with ordered pairs describing points in the plane). Let $P = (x, y)$ and let $Q = (x', y')$ such that $P < Q$. We describe the open interval (P, Q). If $x < x'$, then we get the open ray consisting of all points directly above P, as well as all the points (w, z) with $x < w < x'$ (an infinite open strip) and the open ray consisting of all points directly below Q. If $x = x'$, then we get a vertical open interval connecting the two points P and Q.

We have described the order topology on the ordered set X using a basis, but we can also describe it using a sub-basis.

Definition Let X be an ordered set, let $a \in X$. Then a determines four rays:

$$(a, +\infty) = \{x \in X : a < x\}$$
$$(-\infty, a) = \{x \in X : x < a\}$$
$$[a, +\infty) = \{x \in X : a \leq x\}$$
$$(-\infty, a] = \{x \in X : x \leq x\}$$

The first two are the open rays determined by a, the second two are the closed rays determined by a.

Theorem 2. *Let X be an ordered set. Then the collection of all open rays determined by elements of X is a sub-basis for the order topology. More specifically, every basis element in the order topology is a finite intersection of rays.*

The Product Topology

Definition Let X and Y be topological spaces. The collection $\mathcal{B}$ of all sets $U \times V$, where $U \subset X$ and $V \subset Y$ are open, forms the basis of topology on $X \times Y$ called the *product topology*.

We check that $\mathcal{B}$ is actually a basis. As X and Y are open subsets of themselves, the pair (x, y) is contained in the basis element $X \times Y$. Suppose that $(x, y) \in U_1 \times V_1 \cap U_2 \times V_2$ for $U_1, U_2 \subset X$ open and $V_1, V_2 \subset Y$ open. As $U_1 \times V_1 = \{(x, y) : x \in U_1, y \in V_1\}$, we have $x \in U_1$ and $x \in U_2$, so that $x \in U_1 \cap U_2$. Similarly $y \in V_1 \cap V_2$. Then the basis element $U_1 \cap U_2 \times V_1 \cap V_2$ contains (x, y). You can check that $U_1 \cap U_2 \times V_1 \cap V_2 \subset U_1 \times V_1 \cap U_2 \times V_2$.

Theorem 3. *Let $\mathcal{B}$ be a basis for the topology of X and let $\mathcal{C}$ be a basis for the topology of Y. Then $\mathcal{D} = \{B \times C : B \in \mathcal{B}$ and $C \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.*

Proof. As basis elements are open, and the product $U \times V$ of open sets are open in the product topology, the sets in $\mathcal{D}$ are ll Let $W \subset X \times Y$ be open and let $(x, y) \in W$. To show that $\mathcal{D}$ is a basis for the topology, we have to show that there exists $B \times C \in \mathcal{D}$ such that $(x, y) \in B \times C \subset W$. As W is open in the product topology, we can find a product of open sets $U \times V$ such that $x \in U \times V \subset W$. Now U and V are open in X and Y respectively, so we can find $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subset U$ and $y \in C \subset V$. Thus $(x, y) \in B \times C \subset U \times V$. $\square$

Definition Let $\pi_1 : X \times Y \to X$ be given by $\pi_1(x, y) = x$, and $\pi_2 : X \times Y \to Y$ given by $\pi_2(x, y) = y$.

Theorem 4. *The collection $\{\pi_1^{-1}(U) : U \subset X$ open$\} \cup \{\pi_2^{-1}(V) : V \subset Y$ is open$\}$ is a subbasis for the product topology on $X \times Y$.*

Subspace Topology

Let (X, τ) be a topological space and let $Y \subset X$. Now we consider a natural way to define a topology σ on Y.

Definition If (X, τ) is a topological space and $Y \subset X$, we define the *subspace topology* on Y as follows: a set $U \subset Y$ is open in the subspace topology on X if there is an open set $V \subset X$ such that $V = Y \cap U$. We also call this the topology that Y inherits from X.
Let's prove that the subspace topology is actually a topology on Y:

(1) First, we check that Y and $\emptyset$ are open in the subspaee topology. $Y = X \cap Y$ and $\emptyset = Y \cap \emptyset$.
(2) Now we check that a union of open subsets in the subspace topology is again open in Y. Let $\{U_i\}_{i \in I}$ be a collection of open sets in Y. Then for each U_i there is an open set V_i (in τ) such that $U_i = Y \cap V_i$. Then $V = \cup_{i \in I} V_i$ is open as (X, τ) is a topological space. The distributive law gives us
$$\cup_{i \in I} U_i = \cup_{i \in I} Y \cap V_i = Y \cap \cup_{i \in I} V_i = Y \cap V.$$
As the latter set is open in Y in the subspace topology, the union $\cup_{i \in I} U_i$ is open in Y.
(3) Now let's prove that the intersection of finitely many open sets in the subspace topology is also open. Let $U_1, \ldots, U_n$ be a collection of subsets of Y which are open in the subspace topology. Then $U_i = Y \cap V_i$ for some open sets V_i in X. The intersection
$$U_1 \cap U_2 \cap \ldots \cap U_n = (Y \cap V_1) \cap (Y \cap V_2) \ldots \cap (Y \cap V_n).$$

But we can rewrite this with one instance of Y as follows:

$$Y \cap V_1 \cap V_2 \ldots \cap V_n$$

and as $V_1 \cap V_2 \ldots \cap V_n$ is open in X, the set $Y \cap V_1 \ldots \cap V_n$ is open in Y in the subspace topology. Thus $U_1 \cap U_2 \cap \ldots \cap U_n$ is open in Y in the subspace topology.

Alright, let's see an example of this. **Example** Let $X = \mathbb{R}$ with the standard topology, and let $Y = [0, 1] \subset X$. I claim that the subset $[0, 1/2)$ is open in Y in the subspace topology. For the set $(-1/2, 1/2)$ is open in X, and $[0, 1/2) = (-1/2, 1/2) \cap [0, 1]$. Note that $[0, 1/2)$ is not open in X in the standard topology – it is only open considered as a subset of Y. **Example** The order topology on $\mathbb{N}$ is the same as the subspace topology that $\mathbb{N}$ inherits from $\mathbb{R}$, because $\{n\} = (n-1, n+1) \cap \mathbb{N}$ for every n. Thus the subspace topology on $\mathbb{N}$ is discrete, even though the topology on $\mathbb{R}$ is not discrete. **Example** If $Z \subset Y \subset X$, and X is a topological space, then Y becomes a topological space with the subspace topology. Then we can consider Z as a subspace of Y, with open sets of the form $Z \cap U$ for U open in Y. We can also consider Z as a subspace of X, with open sets of the form $Z \cap V$, for V open in X. These are the same topologies: $W \subset Z$ is open in the subspace topology inherited from Y if and only if $W = U \cap Z$ for some set U which is open in the subspace topology on Y if and only if $W = V \cap Y \cap Z$ for some open set V in X if and only if $W = V \cap Z$ for open $W \subset X$ (as $Y \cap Z = Z$).