MATH 54 - LECTURE 5

DAN CRYTSER

Introduction

In this lecture we introduce the order and product topologies as examples of topologies generated by bases. We also introduce the subspace topology on any subset of a topological space.

Order Topology

Definition Let X be an ordered set with more than one element. Then the collection of sets of the form

$$(a, b)$$

$$[a_0, b) \text{ where } a_0 \text{ is the smallest element of } X \text{ if such exists}$$

$$(a, b_0] \text{ where } b_0 \text{ is the largest element of } X \text{ if such exists}$$

is a basis, and it generates a topology on X known as the *order topology.* **Example** The order topology on $\mathbb{R}$ is just the standard Euclidean topology. The basis for the order topology consists solely of open intervals, as there is no smallest or largest element. But this is the same as the basis for the Euclidean topology, which consists of open balls. As the two bases are the same, the topology that they generate is the same.

Example The order topology on $\mathbb{N}$ is the discrete topology. For if $n > 1$ is an integer, we have $\{n\} = (n-1, n+1)$, and if $n = 1$, we have $\{1\} = [1, 2)$. These sets are basis elements, hence open in the order topology. But if all the singleton sets are open in the order topology, it must be the discrete topology.

Example The dictionary order on $\mathbb{R} \times \mathbb{R}$ is given by $a \times b < c \times d$ if $a < c$ or if $a = c$ and $b < d$ (using different notation for points in the plane to distinguish them from open intervals). Claim: the order topology on $\mathbb{R} \times \mathbb{R}$ is strictly finer than the Euclidean topology. Because we have that the open ball of radius ϵ about the point $a \times b$ is open in the order topology – namely that the interval $(a \times b - \epsilon, a \times b + \epsilon)$ in $\mathbb{R} \times \mathbb{R}$ is contained in the ball and contains $a \times b$. As we can find a basis element for the order topology containing a given point and contained within the given ball, the open ball is open in the order topology (note: we have taken the point in question to be the center of the ball. Technically we need to pick an arbitrary point of the ball, but we can do that and then center another open ball around this arbitrary point so that the second ball is contained within the first, as in the proof that the open balls are open sets in $\mathbb{R}^n$). The y-axis is open in the order topology, being the union of the intervals $(0 \times k, 0 \times k + 2)$ for k ranging across all $\mathbb{Z}$ the set of integers.

Definition Suppose that X and Y are topological spaces. The product topology on $X \times Y$ is generated by the basis $\mathcal{B} = \{U \times V : U \subset X \text{ is open }, V \subset Y \text{ is open }\}$.
Note that this definition asserts without proof that $\mathcal{B}$ is a basis. You can show that $(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$, i.e. the intersection of the products is the product of the intersections. This along with the fact that $X \times Y \in \mathcal{B}$ implies that $\mathcal{B}$ is a basis.

Theorem 1. *Let $\mathcal{B}$ be basis generating the topology on the space X, and similarly let $\mathcal{C}$ be a basis generating the topology on the space Y. Then $\mathcal{D} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis generating the product topology on $X \times Y$.*

Proof. The proof that $\mathcal{D}$ is a basis is almost the same as showing that the collection in the definition of the product topology is a basis.

If $(x, y) \in B_1 \times C_1 \cap B_2 \times C_2$, then $x \in B_1 \cap B_2$ and $y \in C_1 \cap C_2$. Then as $\mathcal{B}$ and $\mathcal{C}$ are bases, find $B_3 \in \mathcal{B}$ and $C_3 \in \mathcal{C}$ such that $x \in B_3 \subset B_1 \cap B_2$ and $y \in C_3 \subset C_1 \cap C_2$, and note that $(x, y) \in B_3 \times C_3 \subset (B_1 \times C_1) \cap (B_2 \times C_2)$.

All the sets in $\mathcal{D}$ are basis elements in the product topology, hence the topology on $X \times Y$ generated by $\mathcal{D}$ is contained in the product topology. We show that if $W \subset X \times Y$ is open in the product topology, then it is open in the topology generated by $\mathcal{D}$.

Suppose that $(x, y) \in W$. Then as W is open in the product topology, we find $U \subset X$ and $V \subset Y$ open sets such that $(x, y) \in U \times V \subset X \times Y$. Now using the fact that the bases $\mathcal{B}$ and $\mathcal{C}$ generate their respective topologies, find $B \in \mathcal{B}$ and $C \in \mathcal{C}$ such that $x \in B \subset U$ and $y \in C \subset V$. Now $(x, y) \in B \times C \subset U \times V \subset W$, and W is open in the topology generated by $\mathcal{D}$.

□

Lemma 1. *Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases on a set X. Then the following statements are equivalent:*

1. *$\mathcal{B}$ and $\mathcal{B}'$ generate the same topology on the set X.*
2. *For every $B \in \mathcal{B}$ and $x \in B$, there exists $B' \in \mathcal{B}'$ such that $x \in B' \subset B$, and vice versa. That is, the basis elements are open in one another's topologies.*

Combining the previous two results allows us to show that **Example** The standard topology on $\mathbb{R}^2$ and the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (where each factor comes equipped with the standard topology) are the same topology.

Proof: We just need to show that open rectangles, being the basis elements for the product topology, are open in the standard topology, and that open balls, being the basis elements in the standard topology, are open in the product topology.

You can see this by considering Figure 13.4 on page 81 of Munkres.
NOTE: As we started by defining the standard topology on $\mathbb{R}^2$ to be the topology given by the basis of open balls, we seem at first to disagree with Munkres' use of the term "standard" to denote the product topology on $\mathbb{R} \times \mathbb{R}$. But this proof shows that there is no problem with using the same term for each – they are the same topology.

Definition Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be given by $\pi(x, y) = x$ and $\pi_2(x, y) = y$. These are the (canonical) projection maps.

We did not prove in class, but it is not overly complicated to prove that:

Theorem 2. *Let X and Y be top. spaces. Then the product topology on $X \times Y$ is generated by the sub-basis* $\{\pi_1^{-1}(U) : U \subset X \text{ open }\} \cup \{\pi_2^{-1}(V) : V \subset Y \text{ open }\}$.

This is because $U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$.

The Subspace Topology

Definition Let X be a top. space and let $Y \subset X$. We define the subspace topology on Y as follows: if $U \subset Y$, then U is open if and only if there is an open set $V \subset X$ with $U = V \cap Y$. (V is open in the topology of X).

It follows from the arbitrary case of the distribution laws that you proved in the first homework assignment that this defines a topology on Y.

Example Let $X = \mathbb{R}$ have the standard topology and $Y = [0, 1]$. Give Y the subspace topology.

(1) The set $[0, 1]$ is not open as a subset of X, but it is open as a subset of Y.
(2) The sets $(0, 1]$ and $[0, 1)$ are open in Y but not in X, as you can take $V = (0, 2)$ and $V = (-1, 1)$ respectively.
(3) The set $(0, 1)$ is open in both X and Y.